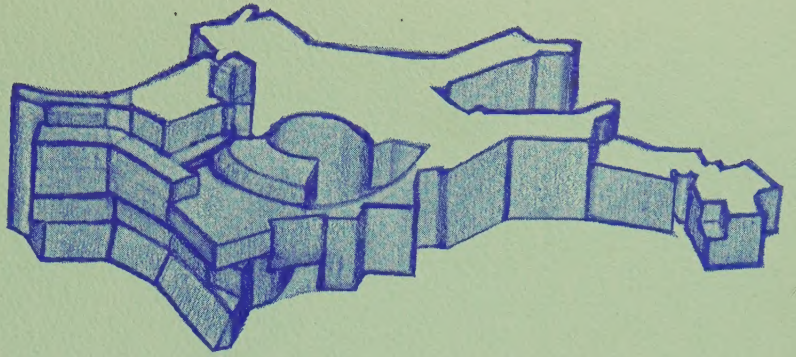


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An Approximation Method for the Determination of the
Motion of Extended Bodies in General Relativity

(Eine Näherungsmethode für die Bestimmung der Bewegung
ausgedehnter Körper in der Allgemeinen Relativitätstheorie)

Dissertation der Fakultät für Physik der
Ludwig-Maximilians-Universität München

vorgelegt von
Alessandro Caporali
aus Padova (Italien)

München, den 18. September 1979

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Max-Planck-Institut für Physik und Astrophysik, München.

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1. Gutachter: Prof. J. Ehlers

2. Gutachter: Prof. F. Schmeidler

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Einleitung

Fast alle quantitativen Informationen über die Gravitationswechselwirkung hat man von isolierten Systemen, wie z.B. Planeten-Systemen oder Doppelstern-Systemen, bekommen.

Schon zu Beginn der Allgemeinen Relativitätstheorie hat man versucht, formelle analytische Näherungsmethoden zur relativistischen Beschreibung isolierter N-Körper Systeme zu entwickeln. Im Jahre 1977 hat Ehlers die grundlegenden Ideen für ein neues Verfahren angegeben, das besonders geeignet ist, die Einstein'sche Gleichung im Fall von realistischen astronomischen Systemen näherungsweise zu lösen. Dieses Verfahren hat mir die Möglichkeit gegeben, das Problem der relativistischen Bewegung eines ausgedehnten Körpers in einem isolierten N-Körper von einem modernen Standpunkt aus zu betrachten.

Das Program dieser Arbeit ist folgendes:

- 1) Das Näherungsverfahren zu benutzen, um eine Lösung und gleichzeitig die entsprechenden lokalen Bewegungsgleichungen für die Quelle zu bekommen.
- 2) Eine Methode zu finden, um aus den so erhaltenen lokalen Bewegungsgleichungen Gleichungen für die Weltlinie und, falls es möglich ist, für den Drehimpuls jedes Körpers des Systems zu erhalten. Bekannterweise läßt sich ja für die meisten Zwecke der Himmelmechanik ein Körper durch eine Weltlinie und durch die ersten Multipol-Momente beschreiben.
- 3) Die auf diese Weise erhaltenen Gleichungen für die Bewegung eines jeden Körpers als Ganzen zu lösen und die Lösung mit beobachtbaren Größen zu verknüpfen.

Kapitel 1 fängt mit einer kurzen Beschreibung des von Ehlers vorgeschlagenen Verfahrens an. Korrekturen erster und zweiter Ordnung zur flachen Minkowski'schen Metrik werden durch Iteration einer "reduzierten" Einstein'schen Gleichung berechnet. Die Metrik, die die Korrektur erster Ordnung enthält, entspricht der Newton'schen Theorie in ihrer vierdimensionalen Formulierung; die Metrik, die

dazu die Korrekturen zweiter Ordnung enthält, wird als Metrik der ersten nach-newtonschen Näherung (1PNA) bezeichnet. Unter milden Voraussetzungen über die Stärke des Gravitationsfeldes haben die beiden Metriken Lorentz'sche Signatur. Nach bestimmten Regeln für die Fehlerabschätzung im Iterationsverfahren wird nachgewiesen, daß die beiden Metriken nicht nur den reduzierten Feldgleichungen entsprechen, sondern auch der Einstein'schen Gleichung, wenn die lokalen Bewegungsgleichungen erfüllt sind. In Kapitel 2 werden im Rahmen der 1. nach-newtonschen Näherung die Masse eines Körpers in einem N-Körper System und die Masse des ganzen Systems definiert. Ferner werden die zugehörigen Bewegungsgleichungen angegeben. Die Masse eines Körpers besteht aus der Ruhemasse, d.h. der Summe der Massen der Baryonen im Körper, und der Summe ihrer nicht relativistischen Gravitations- und ihrer kinetischen Selbstenergie. Diese Masse wird als träge Masse bezeichnet. Genauso enthält die Masse des Systems die Summe der trägen Massen der Körper und ihre nicht-relativistische Selbstenergie. Aus den lokalen Bewegungsgleichungen in der 1PNA folgt, daß die so definierte träge Masse für jeden Körper eine Konstante ist, falls der Körper sich wenig von einer Kugel unterscheidet, oder falls das Verhältnis (Radius des Körpers)/(Abstand von den anderen Körpern) $\ll 1$ ist. Die Eigenschaft der gemeinsamen Masse des N-Körper Systems, eine Konstante zu sein, ist jedoch von der Struktur der Körper und von ihrem gegenseitigen Abstand unabhängig.

In Kapitel 3 werden für jeden Körper des Systems ein nach-newtonscher 3-Impuls $\vec{P}$ und eine nach-newtonsche Kraft $\vec{F}$ definiert. Als Konsequenz der 1PNA Bewegungsgleichungen wird bewiesen, daß dieser Impuls jedes Körpers eine Gleichung der Art

$$\frac{d\vec{P}}{dt} = \vec{F}$$

erfüllt. Falls jeder Körper kugelsymmetrisch ist und nicht rotiert, dann erfüllt seine Weltlinie dieselbe Bewegungsgleichung wie die Einstein-Infeld-Hoffmann'schen Feldsingularitäten. Daher ist im Fall eines gebundenen Zwei-Körper-Systems die Bahn eine sich drehende Ellipse.

Im Kapitel 4 werden die Definitionen und Ergebnisse von Kapitel 3 so verallgemeinert, daß sie einerseits auch andere metrische Gravitationstheorien, z.B. die Brans-Dicke-Jordan'sche "Skalar-Tensor"-Theorie, einschließen, und andererseits von den Körpern nur voraussetzen, daß sie starr rotierende Sphäroide seien. Im Fall eines gebundenen Zwei-Körper-Systems wird gezeigt, wie die Eigenrotation der Körper relativistisch bedingte Änderungen der Bahn verursacht. Darf, als Grenzfall, einer der Körper als Test-Teilchen betrachtet werden, dann stimmen die so erhaltenen Effekte mit den Lense-Thirring'schen Effekten überein.

Die Definition eines nach-newtonschen Eigendrehimpulses wird angegeben. Unter bestimmten Voraussetzungen wird bewiesen, daß der so definierte Eigendrehimpuls jedes der beiden Körper eine relativistische Präzession um die Richtung des Bahndrehimpulses erfährt.

Das Kapitel (und damit die Arbeit) endet mit einigen Bemerkungen über mögliche Anwendungen der in dieser Arbeit entwickelten Näherungsmethode auf die Interpretation der Bewegung des Binär-Pulsars PSR 1913+16.

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An Approximation Method for the Determination of the Motion
of Extended Bodies in General Relativity

Chapter I

A Reformulation of the Post-Newtonian Approximation to General Relativity

Abstract: A new form of a "first post-newtonian approximation to General Relativity" is presented, according to a general iteration scheme for solving Einstein's equation for isolated, self gravitating systems, recently formulated by Ehlers. The post-newtonian metric is constructed and the corresponding equations of motion are set up in two iteration steps. The computation is based on a reduced field equation which is obtained from the full equation by dropping some terms which vanish in harmonic coordinates. The equivalence of the harmonicity condition for the coordinates to the local equations of motion is proved in the post-newtonian approximation. It is shown that the full Einstein equation is approximately satisfied provided the local equations of motion are. In the appendix, a method with rigorous error estimates is developed, in order to obtain the components of the metric tensor g_{ab} , once

$$R^{ab} := \eta^{ab} - \sqrt{-g} g^{ab} \quad (g := \det(g_{ab}))$$

are known.

(0): Introduction and outlook

The improvement of measurements within the solar system to a precision of one part in 10^{11} on the one hand [1], and the discovery of highly relativistic systems like the pulsar PSR 1913+16 in a binary configuration on the other [10], confront gravitational theorists with the problem of making precise predictions about the motion of heavenly bodies. In spite of many efforts, the task of developing, within General Relativity, a physically adequate and mathematically acceptable analog of newtonian celestial mechanics has not yet been mastered. In the elementary derivation of the famous perihelion shift of Mercury's orbit, Mercury is treated as a test particle in the field of the Sun, and the relativistic perturbation is simply added to the newtonian ones. A more realistic and coherent treatment ought to consider several extended, interacting bodies in a relativistic manner, without recourse to either newtonian mechanics or the geodesic hypothesis. Approximate treatments of this kind have been given, but they all suffer from some shortcomings [6]. It is rather clear that "mechanical equations of motion", i.e. ordinary differential equations for finitely many variables describing the bodies of a system, or even finite systems of delay differential equations, can be expected to hold at best approximately and under special conditions in a relativistic field theory. The unsolved problem of specifying such conditions has so far always been bypassed; instead the existence of "equations of motion" has been assumed.

The difficulties one is faced with when one tries to construct relativistic isolated systems have often been emphasized, e.g. by Ehlers (see, e.g. [5] sect. 6) and Geroch [9], and depend mainly on the non integrability of the affine connection

of the curved space-time manifold, and on the non-linearity of Einstein's field equation.

Due to the non integrability of the connection, the integration of vectors and tensors over spacelike hypersurfaces requires a transport law of the integrand function to the integration point, and the covariant equations of motion

$$T^{ab}{}_{;b} = 0 \quad (0.1)$$

do not express in general any conservation law. Another important property of Einstein's theory is that the local equations of motion (0.1) are implied by the field equations, while in newtonian theory and in electrodynamics the equations of motion are independent of them. This means that in any solution method of Einstein's equations, the validity of the equations of motion has to be required as integrability condition, and they are not a problem to be affronted separately. So far, the only covariant description of extended bodies in terms of global variables like center of mass, total 4-momentum etc. in General Relativity is that given by W.G. Dixon [4]. It has recently been extended and improved in [13] and [8] (See also [5] sect. 6 for review). This description does not require the specification of the field equations. Dixon has shown that one can describe bodies in terms of variables which are independent of any space-time geometry: 4-momentum, angular momentum, and an "Energy momentum skeleton" in which only the matter data, independent of the gravitational field, are contained. Dixon's approach has so far not been combined with an approximation method for solving the gravitational field equations, probably because of the complexity of the geometrical constructions by means of which this author has defined variables like 4-momentum, angular momentum etc. of

bodies. At present, one reasonable way to proceed appears to be the patient, stepwise improvement of formal approximation methods, keeping in mind the goal of making contact with those parts of the theory which can be handled rigorously, like Dixon's theory or the asymptotic description of fields near infinity. To the theoretician who, in spite of these difficulties, wants to derive predictions from General Relativity to compare with high precision observations, the development of an approximation method for dealing with field equations and equations of motion appears as the only possibility, just as classical celestial mechanics makes wide use of approximation methods like series expansions and iterations.

So far, no mathematically sound approximation method - i.e. one with proved error-estimates - for the construction of isolated, gravitating systems according to General Relativity has been devised. The best one has done so far was to develop in analogy with simpler theories (classical mechanics, electrodynamics) formal algorithms which might, according to plausibility arguments, give approximate solutions of Einstein's equation. In particular, in his 1977 Erice Lecture-Notes [7], Ehlers has proposed a definition of an isolated gravitating system; by modifying and extending previous work of Fock, Levi-Civita, Papapetrou, Hennequin, Trautman, Chandrasekhar and associates, Synge, Anderson and Decanio ..., he has outlined a systematic method for obtaining, for a certain class of energy-momentum tensors T^{ab} , a sequence g_{ab} of metrics and a corresponding sequence of approximate equations of motion, such that, if $({}_N g_{ab}, {}_N T^{ab})$ satisfy the appropriate approximate equations of motion, then, according to certain "plausible" formal rules for estimating errors - rules which, however, have not been rigorously justified - those fields satisfy approximately Einstein's equation

$$R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} \quad (0.2)$$

The present investigation is based on Ehlers' method which is used in order to obtain local equations of motion (0.1) in a post-newtonian approximation. For this purpose, in sect. 1, the structure and properties of the equations we want to solve are discussed.

In sect. 2 Ehlers' method is schematically exposed. In sect. 3 the general solution method is adapted to the assumptions of "slow motion, weak field" (which are characteristic properties of a large number of astronomical systems), and the first iteration to solve the "reduced" (i.e. referred to a harmonic coordinate system) Einstein's equation is carried out. The second iteration is made in sect. 4, leading to the post-newtonian metric. Further, it is proved that the harmonic coordinate condition and the local equations of motion are approximately equivalent to each other (Theorem 1), and that the approximate solution ^{is also a solution of} of the "full" equation (0.2), modulo the harmonic coordinate condition (Theorem 2). We conclude with sect. 5, where the local equations and uniqueness of their analytic solutions is discussed. In the appendix we compute the contra- and co-variant components of the metric in terms of the quantities

$$R^{ab} := \eta^{ab} - g^{ab}$$

where

$$g^{ab} := \sqrt{-g} \, g^{ab} \quad (g := \det(g_{ab}))$$

which turn out to be more useful as unknown functions than the g 's.

Notations and Conventions: we will basically adhere to [7]. The metric will have signature $(- + + +)$, and units and dimensions are chosen such that

$$8\pi G = 1 = c$$

We also take as unit of length the radius of the most compact body, so that all quantities are dimensionless.

Latin indices range from 0 to 3, greek ones from 1 to 3, unless otherwise stated. Covariant derivation is expressed by a semi-colon.

Conventions on the definitions of the Riemann and Ricci tensors are the ones of [11], ch. 8.

We will use the symbol

$$[\dots]$$

to denote the orders of magnitude.

An equation

$$A = [\varepsilon^4, \varepsilon^2 \delta^2]$$

means that the absolute value of A is pointwise roughly of the order of, and not greater than

$$\max(\varepsilon^4, \varepsilon^2 \delta^2)$$

where ε, δ are small, dimensionless parameters to be defined in sect. 3.

(1): Remarks on the structure of Einstein's equation

Einstein's equation

$$R_{ab} - \frac{1}{2} g_{ab} R = T_{ab} \quad (1.1)$$

consists of ten coupled, second order, quasi-linear differential equations for the components of the metric tensor g_{ab} and the matter tensor. As usual, R_{ab} is the Ricci tensor, obtained from the contraction of the Riemann tensor:

$$R_{ab} = R^m{}_{amb}$$

R is the trace of R_{ab} and T_{ab} is the energy-momentum tensor, which plays the role of a source term, and will be assumed hereafter to be the one of a perfect fluid.

Eq. (1.1) implies the local equation of motion

$$T^{ab}{}_{;b} = 0 \quad (1.2)$$

because of the contracted Bianchi identities

$$G^{ab}{}_{;b} = 0 \quad (1.3)$$

where

$$G^{ab} := R^{ab} - \frac{1}{2} g^{ab} R \quad (1.4)$$

is the Einstein tensor.

Defining

$$g := \det(g_{ab}) \quad (1.5)$$

and

$$\mathcal{g}^{ab} := \sqrt{-g} g^{ab} \quad (1.6)$$

the Einstein equation takes, in the harmonic gauge

$$\mathcal{g}^{ab}_{;b} = 0 \quad (1.7)$$

the form

$$\sqrt{-g} \square_g \mathcal{g}^{ab} = 2 U^{ab} + \mathcal{g}^{ac}_{,d} \mathcal{g}^{bd}_{,c} \quad (1.8)$$

where

$$\square_g := (-g)^{1/2} \partial_a (\mathcal{g}^{ab} \partial_b) \quad (1.9)$$

is the wave operator for scalar fields associated with g_{ab} , and

$$U^{ab} := (-g) (T^{ab} + t^{ab})$$

is the Landau Lifschitz's complex, with t^{ab} the Landau Lifschitz energy momentum pseudotensor of the gravitational field.

Equation (1.8) is called the reduced Einstein equation, irrespective of whether the coordinates satisfy the harmonicity condition (1.7).

Inspection of the principal part of (1.8) reveals that this system is hyperbolic, provided that the metric is lorentzian. For a given set of initial data, and certain forms of the energy-momentum tensor T^{ab} (e.g. a perfect fluid), existence and uniqueness of the solution of (1.2) and (1.8) have been proved, as well as the equivalence, modulo a diffeomorphism, of the solution with any other solution having the same Cauchy data, provided that on a spacelike hypersurface $t = 0$ the following conditions are satisfied (Choquet-Bruhat, in [14])

$$\gamma^{ab}_{,b} = 0 \quad ; \quad \gamma^{ab}_{,bt} = 0 \quad \text{at } t = 0$$

that is, the harmonic condition and its first time derivative must be satisfied at $t = 0$.

It should be stressed, however, that the Ehlers method, which will be used in the following, does not use this formulation of the initial value problem, and the associated existence and uniqueness theorems. In fact, in analogy with the Liénard-Wiechert retarded solutions of Maxwell's equations, the field γ^{ab} , with the use of the Fock's "no incoming radiation" condition, will be expressed as a retarded integral over an "effective source" containing also the second derivatives of γ itself. Hence this relation represents no solution of (1.8), but reformulates (1.8) in the form of an integro differential equation. In this case one is not then faced with an ordinary Cauchy problem, where one puts data at $t = 0$ and looks at their evolution over a finite lapse of time, but rather with an "asymptotic characteristic initial value problem", where the field at a certain time is determined by the values of matter and fields from the infinite past up to that instant of time.

(2): Iterative solution of the Einstein equation

The method devised by Anderson [2], and modified and in part justified by Ehlers [7], in order to obtain approximate solutions of Einstein's equation, proceeds in the following steps:

- a) insert into Einstein's equation

$$g^{ab} = \eta^{ab} - R^{ab} \quad (2.1)$$

- b) formulate that equation in harmonic coordinates [see (1.8)], substitute the d'alambertian operator $\square_g$ of the "true" metric with the one, $\square_\eta$, corresponding to the auxiliary flat metric, and put the difference $(\square_\eta - \square_g)g^{ab}$ on the right hand side:

$$\frac{1}{2} \square_\eta R^{ab} = -U^{ab} + (R^a [{}^b R^c]{}^d)_{,cd} \quad (2.2)$$

where use has been made of (2.1) (see also eq. 12 in [7])

- c) assume Fock's condition for absence of incoming radiation, and convert the reduced Einstein equation (2.2) into an integro-differential equation (eq. (2.3) below). For conditions under which its solutions also solve (1.1) see eq. 16 in [7]
- d) starting from a known solution (e.g. $k_0=0$, corresponding to the Minkowski metric) generate by iteration a sequence ${}_1k, {}_2k, \dots, {}_Nk$ of metric deviations, and compute from them metrics ${}_1g, {}_2g, \dots, {}_Ng$ (see Appendix)

e) Impose the N-th order approximate local equations of motion on the pair $({}_N g_{ab}, {}_N T^{ab})$, where

$${}_N T^{ab} := T^{ab} [\text{matter variables}, {}_N k^{ab}].$$

For a given model of the material source (e.g. perfect fluid), the energy momentum tensor is in fact a functional of the matter variables (density, velocity, pressure) and of the metric tensor.

The method has been extensively discussed in [7] and will not be reviewed here. Rather we stress some steps in the logical chain of the procedure, which we believe to be particularly critical and in need of further investigation.

First, if (g_{ab}, T^{ab}) is a solution of (1.1) on R^4 which permits global harmonic coordinates, and if the functions k^{ab} (see (2.1)) as well as T^{ab} decrease sufficiently fast or past null cones of the flat, auxiliary metric η_{ab} , then (see propositions 1 and 2 in [7], sect. 3) k^{ab} obeys the integro-differential equation

$$R^{ab}(x, t) = \int_{N_x^-} V^{ab}(x + y, t - |y|) \frac{d^3 y}{|y|} \quad (2.3)$$

where

$$2\pi V^{ab} := U^{ab} - (R^{a[b} R^{c]d})_{,cd} \quad (2.3a)$$

and N_x^- is the past null cone of the field point (x, t) , with measure $d^3 y / |y|$. It is important to point out that the conversion (2.2) $\rightarrow$ (2.3) is based on the assumption that Fock's condition on the fall-off of k along the past null cone of the flat metric η :

$$R^{ab} = o(1) \quad ; \quad \left(\frac{\partial}{\partial t} + \frac{\partial}{\partial r} \right) R^{ab} = o\left(\frac{1}{2}\right)$$

guarantees the absence of incoming radiation. It is not known whether Fock's condition is physically correct, since it is based on the use of the unphysical (flat) null cones, rather than on the use of the light cones belonging to the (unknown) physical metric g .

From the use of the incorrect metric for defining the light cones, it follows that the structure of the past null infinity $\mathcal{I}^-$ based on η should deviate from the one of $\mathcal{I}^-$ (if any!) concerning the full metric g . If the asymptotic structures of (M, g) and (M, η) are basically different near past null infinity, then the method developed in [7] and in the present investigation has to be modified.

Another point which deserves attention is the convergence or asymptotic character of the iteration method by which (2.3) is to be solved. Any meaningful statement here needs an estimate of the error one commits, when the iteration is truncated after a few times. But unfortunately no rigorous error estimate for such iteration method has been formulated. One can then only conjecture that, according to plausible arguments, the iteration provides an (in some sense; see next sections) approximate solution of Einstein's equation.

Finally, a remark on the equations of motion. The complexity of the analytical form of the approximate metric ${}_N g$ grows with N . Since these metric functions have to be used in the local equations of motion of the N -th order, these equations will presumably not appear in a meaningful or manageable form. For the first few values of N the equations of motion can probably

be handled, and maybe existence and uniqueness of their solution can be proved. But for large N the analytical structure of the equation is expected to become prohibitively complicated.

Since these difficulties have not been overcome, we use the method described in [7] as "Poincaré invariant, ("Fast") iteration scheme" as the primary approximation method.

(3): A slow motion approximation

For many astronomical systems the typical velocities are small compared with the speed of light, and also the maximum value of the potential is small, compared to the square of the speed of light. It is therefore meaningful to introduce these special assumptions (slow motion, weak field) in the "Fast approximation scheme" of the previous section, in order to obtain explicit approximate expressions of the metric tensor in a near zone.

According to [7], sect. 5, the assumption of slow motion is introduced into the basic equations (2.3) by expanding in a series of powers the retardation $|y|$ contained in the integrand in the right hand side of (2.3). The modified iteration in solving (2.3) proceeds along with the series expansion, so that at each new iteration further terms in the series expansion are taken into account.

The series expansion of the integrand in (2.3) is

$$\begin{aligned}
 |y|^{-1} \bar{V}(x+y, t-|y|) &= \\
 &= \sum_{\ell=0}^n \frac{(-)^\ell}{\ell!} |y|^{\ell-1} \partial_t^\ell V(x+y, t) + \\
 &+ \frac{(-)^{n+1}}{(n+1)!} |y|^n \partial_t^{n+1} V(x+y, t^*)
 \end{aligned} \tag{3.1}$$

with $t - |y| < t^* < t$, where the second term in the right hand side of (3.1) denotes the rest.

Let

$$I_\ell[f](x, t) := \int f(x, x', t) |x - x'|^\ell d^3x' \tag{3.2}$$

We can insert (3.1) into (2.3) under the assumption of decrease of the time-derivatives of $V(x+y, t)$ (see assumption 3 in [7]).

We obtain:

$$R^{ab}(x, t) = \sum_{l=0}^m \frac{(-)^l}{l!} I_{l-1} \left[\partial_t^l V^{ab} \right] + e_m^{ab} \quad (3.3)$$

The fact that

$$V^{ab}_{,b} = 0 \quad (3.4)$$

implies

$$\begin{cases} \partial_t^{l+2} V^{00} = \partial_t^l V^{\alpha\beta}_{,\alpha\beta} \\ \partial_t^{l+1} V^{0\alpha} = -\partial_t^l V^{\alpha\beta}_{,\beta} \end{cases} \quad (l \geq 0) \quad (3.5)$$

Writing

$$V^{\alpha\beta}(x, x', t) := V^{\alpha\beta}(x', t) (x_\beta - x'_\beta) |x - x'|^{-1} \quad (3.6a)$$

$$V^\perp(x, x', t) := V^{\alpha\beta}(x', t) \left(\delta_{\alpha\beta} - \frac{(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|x - x'|^2} \right) \quad (3.6b)$$

and eliminating the divergence of $V^{\alpha\beta}$ using the Gauss theorem (see [7]), we obtain

$$k := k^{00} = I_{-1} \left[V + \frac{1}{2} V_{,1} \right] + \\ + \sum_{l=1}^n \frac{(-)^l}{l! (l+2)} I_{l-1} \left[\partial_t^l V^{''} + (l-1) V^{'''} \right] + e_{n+2}$$

$$k^\alpha := k^{0\alpha} = I_{-1} \left[V_\alpha \right] + \\ + \sum_{l=1}^n \frac{(-)^l}{(l-1)! (l+1)} I_{l-1} \left[\partial_t^l V^{\alpha''} \right] + e_{n+2}^\alpha$$

$$k^{\alpha\beta} = I_{-1} \left[V^{\alpha\beta} \right] + \\ + \sum_{l=1}^n \frac{(-)^l}{l!} I_{l-1} \left[\partial_t^l V^{\alpha\beta} \right] + e_n^{\alpha\beta} \quad (3.7)$$

where e_n denotes the integral of the rest in the expansion (3.1).

Equation (3.7) are a slow motion version of the general equation (2.3), since instantaneous integrals of V^{ab} and of its time-derivatives have taken the place of the retarded ones.

The iterative solution of (3.7) requires the specification of the material source, and rules for assigning orders of magnitude. The energy momentum tensor T^{ab} has already been assumed to be the one of a perfect fluid, together with an equation of state

$$T^{ab} = [\varrho (1 + \pi) + p] u^a u^b + p g^{ab}, \quad \pi = \pi(\varrho)$$

where ϱ, π, p are the density, the specific internal energy and the pressure, respectively, $\pi = \pi(\varrho)$ is a given equation of state, and

$$u^a = \frac{dx^a}{d\tau} \quad (\tau = \text{proper time})$$

is the four-velocity of the matter, normalized to -1.

The coordinate three-velocity is defined by

$$v^\alpha := u^\alpha / u^0$$

Because of the equation of state $\pi = \pi(\varrho)$, and the restriction to adiabatic processes,

$$d\pi + p d\left(\frac{1}{\varrho}\right) = 0$$

it follows that pressure and density are functionally dependent:

$$p(\varrho) = \varrho^2 \frac{d\pi}{d\varrho}$$

Among the six hydrodynamical variables so far introduced, only four, e.g. ϱ , v_α , are then independent, as in newtonian theory.

Let $\delta \ll 1$ be the typical value of the velocity and ε^2 the typical compactness of the bodies ($\varepsilon^2 \sim \frac{m}{4\pi} \ll 1$, remembering that in our unities the size of the most compact body is the unit length). Orders of magnitude are then assigned according to

$$\varrho = [\varepsilon^2] = \pi \quad ; \quad p = [\varepsilon^4, \varepsilon^2 \delta^2] \quad ; \quad v = [\delta]$$

and, following [7], the formal rules for recognizing orders of magnitude of quantities A, B, ... referring to the system, are

$$\{A+B\} \leq \{A\} + \{B\} \sim \max(\{A\}, \{B\})$$

$$\{AB\} \leq \{A\} \cdot \{B\}$$

(3.8)

$$\{A_{,\alpha}\} \leq \{A\} \quad ; \quad \{A_{,t}\} \leq \delta \{A\}$$

$$\{I_L[A]\} \leq L^L \{A\} \quad L \gg 0$$

The symbol $\{\dots\}$ means "order of magnitude of $\dots$ " and L is the size of the near zone, where the source is assumed to be confined, and such that

$$L \ll \frac{\text{amplitude of the motion}}{\text{typical velocity}} = \text{typical length}$$

Rules (3.8) are assumed in an approximately inertial coordinate system, and, as emphasized in [7] and in sect. 1, they are neither rigorously justified, nor accurate error estimates.

The value of the newtonian potential

$$U(x, t) := \frac{1}{8\pi} I_{-1} [\mathcal{S}]$$

never exceeds the compactness ε^2 , so that, according to the notation introduced in sect. 1

$$U = [\varepsilon^2] \quad \text{or} \quad \{U\} = \varepsilon^2$$

Now the energy momentum tensor and the rules for estimating orders of magnitude have been introduced. The iterative solution of (3.7) can then be started, according to the following strategy:

- a) choose as initial lorentzian metric $g^{ab} = \eta^{ab}$, corresponding to ${}_0k^{ab} = 0$
- b) compute ${}_0T^{ab} = T^{ab}(g, v, {}_0k)$, the Landau-Lifshitz complex ${}_0U^{ab}$ and the quantity ${}_0v^{ab}$ defined in (2.3a)
- c) insert ${}_0v^{ab}$ in (3.7) and compute ${}_1k^{ab}$ (*)
- d) go to b) and repeat the same operations with ${}_1k$ at the place of ${}_0k$, and so on. The sequence

$${}_1k, {}_2k, \dots, {}_Nk.$$

is then generated

- e) stop the iteration at a desired value of N , and impose the coordinate condition (1.7) on ${}_Nk$.

The first iteration will be carried out in the rest of this section, while sect. 4 is dedicated to the second iteration.

The Minkowski's metric η will be used as starting point for the first iteration of (3.7). This means

$${}_0k^{ab} = 0 \tag{3.9}$$

(*) The value of n in (3.7) is fixed at each iteration by (3.8)

From the definition of the matter four-velocity it follows that

$${}_0\mu^0 = 1 + [\delta^2] \quad ; \quad {}_0\mu^\alpha = v_\alpha + [\delta^3]$$

The components of the energy-momentum tensor are then

$${}_0T^{00} = \rho + [\varepsilon^4, \varepsilon^2\delta^2]$$

$${}_0T^{0\alpha} = \rho v_\alpha + [\varepsilon^4\delta, \varepsilon^2\delta^3]$$

$${}_0T^{\alpha\beta} = \rho v_\alpha v_\beta + p \delta_{\alpha\beta} + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^3]$$

Due to the constancy of the zeroth-order metric, no contribution arises from the Landau-Lifshitz t_{ab} :

$${}_0t^{ab} = 0$$

so that, at this order, v^{ab} and T^{ab} coincide:

$$V^{ab} = {}_0T^{ab}$$

The iteration of (3.7) provides immediately ${}_4k^{ab}$. Define first the vector potential as the linear functional of ρv_α

$$W_\alpha(x, t) := \frac{1}{8\pi} \int (\rho v_\alpha)' |x - x'|^{-1} d^3x'$$

(a prime (') over a quantity means that it is computed at the point x' and at the time t). The result of the first iteration can then be written as follows:

$$K(x, t) = 4U + [\varepsilon^4, \varepsilon^2 \delta^2]$$

$$K^\alpha(x, t) = 4W_\alpha + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (3.11)$$

$$K^{\alpha\beta}(x, t) = \frac{1}{2\pi} \int (\delta V_\alpha V_\beta + \rho \delta_{\alpha\beta})' |x - x'|^{-1} d^3x' + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]$$

Using the method developed in the appendix, it is found that:

$${}_1g^{00} = -(1 + 2U) + [\varepsilon^4, \varepsilon^2 \delta^2]$$

$${}_1g^{0\alpha} = -4W_\alpha + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

$${}_1g^{\alpha\beta} = \delta_{\alpha\beta} (1 - 2U) + [\varepsilon^4, \varepsilon^2 \delta^2]$$

(3.12)

$${}_1g_{00} = -(1 - 2U) + [\varepsilon^4, \varepsilon^2 \delta^2]$$

$${}_1g_{0\alpha} = -4W_\alpha + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

$${}_1g_{\alpha\beta} = \delta_{\alpha\beta} (1 + 2U) + [\varepsilon^4, \varepsilon^2 \delta^2]$$

(3.13)

$$\sqrt{-g} = 1 + 2U + [\varepsilon^4, \varepsilon^2 \delta^2]$$

It then follows that on the stream lines of matter

$$ds^2 = -dt^2(1 - v^2 - 2U + [\varepsilon^4, \varepsilon^2\delta^2]) \quad (3.14)$$

so that

$${}_1\mu^0 = 1 + \frac{1}{2}v^2 + U + [\varepsilon^4, \varepsilon^2\delta^2] \quad (3.15)$$

$${}_1\mu^\alpha = v_\alpha {}_1\mu^0 = v_\alpha \left(1 + \frac{1}{2}v^2 + U\right) + [\varepsilon^4\delta, \varepsilon^2\delta^3] \quad (3.16)$$

Whence the components of ${}_1T^{ab}$ are

$$\begin{aligned} {}_1T^{00} &= \rho(1 + v^2 + 2U + \pi) + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] \\ {}_1T^{0\alpha} &= \rho v_\alpha (1 + v^2 + 2U + \pi + p/\rho) + [\varepsilon^6\delta, \varepsilon^4\delta^3, \varepsilon^2\delta^5] \\ {}_1T^{\alpha\beta} &= \rho v_\alpha v_\beta (1 + v^2 + 2U + \pi + p/\rho) + p(1 - 2U) \delta_{\alpha\beta} + \\ &\quad + [\varepsilon^8, \varepsilon^6\delta^2, \varepsilon^4\delta^4, \varepsilon^2\delta^6] \end{aligned} \quad (3.17)$$

$$\begin{aligned} {}_1T_{00} &= \rho(1 + v^2 + \pi - 2U) + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] \\ {}_1T_{0\alpha} &= -\rho v_\alpha (1 + v^2 + \pi + 2U) + 4\rho W_\alpha + \\ &\quad + [\varepsilon^6\delta, \varepsilon^4\delta^3, \varepsilon^2\delta^5] \end{aligned} \quad (3.18)$$

$$\begin{aligned} {}_1T_{\alpha\beta} &= \rho v_\alpha v_\beta (1 + v^2 + \pi + 2U + p/\rho) + p(1 - 2U) \delta_{\alpha\beta} - \\ &\quad - 4\rho (W_\alpha v_\beta + W_\beta v_\alpha) + [\varepsilon^8, \varepsilon^6\delta^2, \varepsilon^4\delta^4, \varepsilon^2\delta^6] \end{aligned}$$

This can be considered as the end of the first iteration.

(4): Continuation

In this section, the second iteration of (3.7) will be carried out along the same lines as in the previous section, in order to obtain the metric ${}_2g_{ab}$. Once we have an explicit expression for it, in terms of the matter variables φ, v_α , then, according to the method sketched at the beginning of sect. 2, the equations of motion

$$T^{ab}{}_{;b} = 0$$

must be imposed on the pair $({}_2g, {}_2T)$.

It will then be proved (Theorem 1) that the equations of motion are approximately equivalent to the De Donder condition (1.7) on the coordinates (harmonic gauge). Furthermore it will be proved (Theorem 2) that the quantities ${}_2g_{ab}$ (or ${}_2k^{ab}$) solving, in the approximation and in harmonic coordinates, the reduced Einstein equation (3.7), also solve the full Einstein equation (1.1) approximately.

Recalling step c) of the scheme of calculation described in the previous section, the basic quantity needed for the second iteration is ${}_1v^{ab}$, that is ${}_2u^{ab}$, since their difference, a quadratic expression in the ${}_1k$'s including their first and second derivatives (see (2.3 a)), is negligibly small, modulo the rules (3.8), as one easily checks.

Once ${}_1g_{ab}$ (see (3.12)) and ${}_1T^{ab}$ (see (3.17)) are known, then also the corresponding U^{ab} may be computed

$$\begin{aligned} {}_1U^{00} &= \varphi (1 + v^2 + \pi + 6U) - 7(\nabla U)^2 + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] \\ {}_1U^{\alpha 0} &= \varphi (1 + v^2 + \pi + 6U + \rho/\varphi) v_\alpha + 2[3U, {}_1U_{,\alpha} + \\ &\quad + 4U_{,\beta} (W_{\beta,\alpha} - W_{\alpha,\beta})] + [\varepsilon^6\delta, \varepsilon^4\delta^3, \varepsilon^2\delta^5] \end{aligned} \quad (4.1)$$

$$\begin{aligned} {}_1U^{\alpha\beta} &= \varphi v_\alpha v_\beta + \rho \delta_{\alpha\beta} + 2U_{,\alpha} U_{,\beta} - \delta_{\alpha\beta} (\nabla U)^2 + \\ &\quad + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] \end{aligned}$$

In this last formula it should be noted that only the part of ${}_1T^{\alpha\beta}$ which is of the same order as ${}_1t^{\alpha\beta}$ has consistently been considered.

As was pointed out by Anderson and Decanio ([2] sect. 5)

$$2\pi {}_1V^{ab} = {}_1U^{ab}$$

It then follows from (3.7):

$$\begin{aligned} {}_2k(x, t) &= I_{-1} \left[{}_1V^{00} + \frac{1}{2} {}_1V^{\perp} \right] + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] = \\ &= 4U + \frac{1}{2\pi} I_{-1} \left[\rho (V^2 + \pi + 6U) - 7(\nabla U)^2 + \frac{\rho V_{\perp}^2}{2} + \right. \\ &\quad \left. + \rho - (\nabla U \cdot n)^2 \right] + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \end{aligned}$$

$$\begin{aligned} {}_2k^{\alpha} &= I_{-1} \left[{}_1V^{\alpha 0} \right] - \frac{1}{2} I_0 \left[{}_1V^{\alpha \perp}, t \right] + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] = \\ &= 4W_{\alpha} + \frac{I_{-1}}{2\pi} \left[\rho (V^2 + 6U + \pi + P/\rho) V_{\alpha} + 2 \left[3U_{,t} U_{,\alpha} + \right. \right. \\ &\quad \left. \left. + 4U_{,\beta} (W_{\beta,\alpha} - W_{\alpha,\beta}) \right] \right] - \\ &\quad - \frac{1}{4\pi} I_{0,t} \left[\pi_{\beta} \left(\rho V_{\alpha} V_{\beta} + \rho \delta_{\alpha\beta} + 2U_{,\alpha} U_{,\beta} - \delta_{\alpha\beta} (\nabla U)^2 \right) \right] + \\ &\quad + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \end{aligned}$$

$$\begin{aligned} {}_2k^{\alpha\beta} &= I_{-1} \left[{}_1V^{\alpha\beta} \right] + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] = \\ &= \frac{1}{2\pi} I_{-1} \left[\rho V_{\alpha} V_{\beta} + \rho \delta_{\alpha\beta} + 2U_{,\alpha} U_{,\beta} - \delta_{\alpha\beta} (\nabla U)^2 \right] + \\ &\quad + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \end{aligned}$$

The notations:

$$V_{\perp}^2 := V^2 - (\underline{V} \cdot \underline{n})^2 \quad ; \quad n_{\alpha} := \frac{x_{\alpha} - x'_{\alpha}}{|x - x'|}$$

have been used. The following notations will also be used:

$$\begin{aligned} I_{-1}[00] &:= I_{-1} \left[\varrho (V^2 + \pi + 6U) - 7(\nabla U)^2 + \frac{\varrho V_{\perp}^2}{2} + p - (\nabla U \cdot \underline{n})^2 \right] \\ I_{-1}[0\alpha] &:= I_{-1} \left[\varrho (V^2 + \pi + 6U + p/\varrho) V_{\alpha} + 2 \left[3 U_{,t} U_{,\alpha} + \right. \right. \\ &\quad \left. \left. + 4 U_{,\beta} (W_{\beta,\alpha} - W_{\alpha,\beta}) \right] \right] \\ I_0[\alpha\parallel] &:= I_0 \left[(\varrho V_{\alpha} V_{\beta} + p \delta_{\alpha\beta} + 2 U_{,\alpha} U_{,\beta} - \right. \\ &\quad \left. - \delta_{\alpha\beta} (\nabla U)^2) n_{\beta} \right] \\ I_{-1}[\alpha\beta] &:= I_{-1} \left[\varrho V_{\alpha} V_{\beta} + p \delta_{\alpha\beta} + 2 U_{,\alpha} U_{,\beta} - \delta_{\alpha\beta} (\nabla U)^2 \right] \\ I_{-1}[00 \pm \mu\mu] &:= I_{-1}[00] \pm I_{-1}[\mu\mu] \end{aligned} \tag{4.2}$$

By means of these abbreviations ${}_2k^{ab}$ may be rewritten in a more compact form:

$$\begin{aligned} {}_2k(x, t) &= 4U + \frac{1}{2\pi} I_{-1}[00] + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \\ {}_2k^{\alpha}(x, t) &= 4W_{\alpha} + \frac{1}{2\pi} I_{-1}[0\alpha] - \frac{1}{4\pi} I_{0,t}[\alpha\parallel] + [\varepsilon^6, \varepsilon^4 \delta^4, \varepsilon^2 \delta^4] \\ {}_2k^{\alpha\beta}(x, t) &= \frac{1}{2\pi} I_{-1}[\alpha\beta] + [\varepsilon^4 \delta, \varepsilon^4 \delta^2] \end{aligned} \tag{4.3}$$

Using the formulae derived in the Appendix, one also has:

$$\begin{aligned}
 {}_2g^{00} &= -\left(1 + 2U + \frac{1}{4\pi} I_{-1}[00 + \mu\mu] - 2U^2\right) + [\varepsilon^4\delta, \varepsilon^2\delta^3] \\
 {}_2g^{0\alpha} &= -4W_\alpha - \frac{1}{2\pi} I_{-1}[0\alpha] + \frac{1}{4\pi} I_{0,t}[\alpha_{||}] + 8UW_\alpha + \\
 &\quad + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] \\
 {}_2g^{\alpha\beta} &= \delta_{\alpha\beta} \left(1 - 2U - \frac{1}{4\pi} I_{-1}[00 - \mu\mu] + 6U^2\right) - \frac{1}{2\pi} I_{-1}[\alpha\beta] + \\
 &\quad + [\varepsilon^4\delta, \varepsilon^2\delta^3]
 \end{aligned} \tag{4.4}$$

$$\begin{aligned}
 {}_2g_{00} &= -\left(1 - 2U - \frac{1}{4\pi} I_{-1}[00 + \mu\mu] + 6U^2\right) + [\varepsilon^4\delta, \varepsilon^2\delta^3] \\
 {}_2g_{0\alpha} &= -4W_\alpha - \frac{1}{2\pi} I_{-1}[0\alpha] + \frac{1}{4\pi} I_{0,t}[\alpha_{||}] + 8UW_\alpha + \\
 &\quad + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4]
 \end{aligned} \tag{4.5}$$

$$\begin{aligned}
 {}_2g_{\alpha\beta} &= \delta_{\alpha\beta} \left[1 + 2U + \frac{1}{4\pi} I_{-1}[00 - \mu\mu] - 2U^2\right] + \frac{1}{2\pi} I_{-1}[\alpha\beta] + \\
 &\quad + [\varepsilon^4\delta, \varepsilon^2\delta^3]
 \end{aligned}$$

$${}_2g = 1 + 4U + \frac{1}{2\pi} I_{-1}[00 - \mu\mu] + [\varepsilon^4\delta, \varepsilon^2\delta^3]$$

In order to facilitate a comparison of the expressions (4.4), (4.5) of the metric tensor with previous post-newtonian calculations, (see in particular [2], [3]), the following "translation scheme" might be useful.

Notation used here	Notation of Ref. [2]
$\delta_{\alpha\beta} \left(\frac{1}{4\pi} I_{-1} [00 - \mu\mu] - 2 U^2 \right) +$ $+\frac{1}{2\pi} I_{-1} [\alpha\beta]$	$- A_{\alpha\beta}^{(4)}$
$\frac{1}{4\pi} I_{-1} [00 + \mu\mu]$	$4\Phi + 4U^2 - \chi_{,tt}$

We also notice that the need for a change in notation with respect e.g. to [2] must be expected, since, following Ehlers, we have eliminated, before the iteration, some time-derivatives with the help of the equations of motion (see (3.5)), a device not used by Anderson and Decanio. It is precisely because of this reason that the method in [7] leads without divergence to the 2 1/2 post-newtonian approximation, where radiation reaction terms first come into play in the equations of motion.

The metric element corresponding to (4.5) along the stream-lines is

$$ds^2 = - \left\{ 1 - v^2 - 2U - \frac{1}{4\pi} I_{-1} [00 + \mu\mu] + 6U^2 + 8W_\alpha V_\alpha - \right. \\ \left. - 2Uv^2 + [\varepsilon^4 \delta, \varepsilon^2 S^3] \right\} dt^2$$

so that

$$\mu^0 = 1 + \frac{v^2}{2} + U - \frac{3}{2} U^2 + \frac{5}{2} U v^2 + \frac{3}{8} v^4 - 4 W_\alpha V_\alpha + \\ + \frac{1}{8\pi} I_{-1} [00 + \mu\mu] + [\varepsilon^4 \delta, \varepsilon^2 S^3]$$

As it already happened in the first iteration, the entire expressions for ${}_2g_{0\alpha}$ and ${}_2g_{\alpha\beta}$ provided by the iteration are not needed in the construction of ds^2 , since they are multiplied respectively by v_α , $v_\alpha v_\beta$, so that their "effective weight" in the metric element becomes smaller by amounts of order δ , δ^2 respectively.

The components of the energy momentum tensor are:

$${}_2T^{00} = \vartheta \left[1 + v^2 + 2U + \pi (1 + 2U + v^2) + \frac{p}{\vartheta} v^2 + 6Uv^2 - 2U^2 + v^4 - 8W_\alpha v_\alpha + \frac{1}{4\pi} I_{-1} [00 + \mu\mu] \right] + [\varepsilon^1 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5]$$

$${}_2T^{0\alpha} = \vartheta \left[(1 + v^2 + 2U)(1 + \pi + p/\vartheta) + 6Uv^2 - 2U^2 + v^4 - 8W_\gamma v_\gamma + \frac{1}{4\pi} I_{-1} [00 + \mu\mu] \right] v_\alpha - 4pW_\alpha + [\varepsilon^8, \varepsilon^6 \delta^2, \varepsilon^4 \delta^4, \varepsilon^2 \delta^6]$$

$${}_2T^{\alpha\beta} = \vartheta v_\alpha v_\beta \left[(1 + 2U + v^2)(1 + \pi + p/\vartheta) + 6Uv^2 - 2U^2 + v^4 - 8W_\gamma v_\gamma + \frac{1}{4\pi} I_{-1} [00 + \mu\mu] \right] + p\delta_{\alpha\beta} \left[1 - 2U - \frac{1}{4\pi} I_{-1} [00 - \mu\mu] + 6U^2 \right] - p \frac{I_{-1}}{2\pi} [\alpha\beta] + [\varepsilon^8 \delta, \varepsilon^6 \delta^3, \varepsilon^4 \delta^5, \varepsilon^2 \delta^7]$$

This can be considered the end point of the second iteration.

Since the calculation was based on the reduced field equation, we must now require the De Donder condition (1.7). Furthermore, according to the beginning of sect. 2, we must impose the equations of motion on the pair $({}_2g, {}_2T)$. In the following theorem it will be proved that, for $N = 2$, the harmonic condition and the equations of motion are equivalent. More precisely

Theorem 1: if one of the three functions

$${}_2R^{0b}{}_{,b} \quad ; \quad {}_2U^{0b}{}_{,b} \quad ; \quad {}_2T^{0b}{}_{,b}$$

is of the order $[\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4]$, then so are the other two. Also, if one of the three sets of functions

$${}_2R^{\alpha b}{}_{,b} \quad ; \quad {}_2U^{\alpha b}{}_{,b} \quad ; \quad {}_2T^{\alpha b}{}_{,b}$$

is of the order $[\varepsilon^4\delta, \varepsilon^2\delta^3]$, then so are the other two.

Note: the sizes of the error terms in the right hand side are immediately recognized by inspection of (4.1), (4.3).

Proof: From (4.3) and (3.8) one has

$$\begin{aligned} {}_2R_{,t} + {}_2R^{\alpha}{}_{,\alpha} &= 4\mathcal{U}_{,t} + \frac{1}{2\pi} I_{-1,t} [00] + 4\mathcal{W}_{\alpha,\alpha} + \\ &\quad + \frac{1}{2\pi} I_{-1,\alpha} [0\alpha] - \frac{1}{4\pi} I_{0,t\alpha} [\alpha\parallel] \\ &\quad + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] \end{aligned}$$

$${}_2R^{\alpha}{}_{,t} + {}_2R^{\alpha\beta}{}_{,\beta} = 4\mathcal{W}_{\alpha,t} + \frac{1}{2\pi} I_{-1,\beta} [\alpha\beta] + [\varepsilon^4\delta, \varepsilon^2\delta^3] \quad (4.6)$$

Using the Gauss theorem in the same way as in sect. 3, and noting that:

$$I_{0,t\alpha}[\alpha] \equiv 2 I_{-1,t} \left[\frac{\varrho v_1^2}{2} + p - (\nabla U \cdot \underline{n})^2 \right]$$

one obtains the following expressions:

$$\begin{aligned} {}_2k_{,t} + {}_2k^{\alpha}_{, \alpha} = & \frac{1}{2\pi} \partial_t \left\{ I_{-1}[\varrho] + I_{-1}[00] - I_{-1} \left[\frac{\varrho v_1^2}{2} + p - \right. \right. \\ & \left. \left. - (\nabla U \cdot \underline{n})^2 \right\} + \frac{1}{2\pi} \left\{ I_{-1} \left[\partial_{\alpha}(\varrho v_{\alpha}) + \right. \right. \\ & \left. \left. + I_{-1} \left[\partial_{\alpha}(0\alpha) \right] \right\} + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \end{aligned}$$

$$\begin{aligned} {}_2k^{\alpha}_{,t} + {}_2k^{\alpha\beta}_{, \beta} = & \frac{1}{2\pi} \partial_t I_{-1}[\varrho v_{\alpha}] + \frac{1}{2\pi} I_{-1}[\partial_{\beta}(\alpha\beta)] + \\ & + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \end{aligned}$$

The symbols:

$$I_{-1}[\partial_{\beta}(0\alpha)] \quad ; \quad I_{-1}[\partial_{\beta}(\alpha\beta)]$$

mean that the integrands have to be differentiated with respect to x'_p at the point (x', t) .

Recalling the definitions (4.2) of $I_{-1}[00]$, $I_{-1}[0\alpha]$ and $I_{-1}[\alpha\beta]$, and taking also into account the expressions (4.1) for ${}_1U^{ab}$, one has:

$${}_2k_{,t} + {}_2k^{\alpha}_{, \alpha} = \frac{1}{2\pi} I_{-1} \left[{}_1U^{00}_{,t} + {}_1U^{0\alpha}_{, \alpha} \right] + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]$$

$${}_2k^{\alpha}_{,t} + {}_2k^{\alpha\beta}_{, \beta} = \frac{1}{2\pi} I_{-1} \left[{}_1U^{0\alpha}_{,t} + {}_1U^{\alpha\beta}_{, \beta} \right] + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (4.7)$$

Application of the operator $-\frac{1}{2} \nabla^2$ gives

$$-\frac{1}{2} \nabla^2 \left({}_2R_{,t} + {}_2R^\alpha_{, \alpha} \right) = {}_1U^{00}_{,t} + {}_1U^{0\alpha}_{, \alpha} + [\varepsilon^4, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (4.8)$$

$$-\frac{1}{2} \nabla^2 \left({}_2R^\alpha_{, t} + {}_2R^{\alpha\beta}_{, \beta} \right) = {}_1U^{0\alpha}_{, t} + {}_1U^{\alpha\beta}_{, \beta} + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

(4.7) shows that if the equations of motion are satisfied, then the coordinate ^{condition} must hold, while the converse is proved by (4.8).

The equivalence, modulo field equation and errors, of the two versions of the equations of motion

$$U^{ab}_{,b} = 0 \quad \text{and} \quad T^{ab}_{;b} = 0$$

is, at this level of approximation, a well established statement, and the proof will not be repeated here. This completes the proof of Theorem 1.

The final question to be discussed is the following: the metric g given in (4.4), (4.5) is the result of a twofold iteration based on the reduced Einstein equation combined with the slow motion assumptions (3.7). It must therefore be checked whether the "full" Einstein equation

$$R_{ab} = T_{ab} - \frac{1}{2} g_{ab} T$$

is also solved approximately by (4.5), if the coordinates are harmonic to the stated order of approximation.

To give a meaning to this statement, we define the error-tensor by

$${}_N E_{ab} = {}_N R_{ab} - \left({}_N T_{ab} - \frac{1}{2} {}_N g_{ab} {}_N T \right) \quad (N=1, 2 \dots)$$

We now state the

Theorem 2: the approximate harmonicity conditions

$${}_2k^{ab},{}_b = [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4]$$

$${}_2k^{\alpha b},{}_b = [\varepsilon^4\delta, \varepsilon^2\delta^3]$$

are equivalent to

$${}_2E_{00} = [\varepsilon^4\delta, \varepsilon^2\delta^3] \quad ; \quad {}_2E_{\alpha\beta} = [\varepsilon^4\delta, \varepsilon^2\delta^3]$$

$${}_2E_{0\alpha} = [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4]$$

Note: in the definition of the error tensor, the index of iteration appears with the different values N and $N-1$. The reason lies in the notation we have used, and also in the logic of the things: in the formulation (3.7) (or (2.3)) of the reduced Einstein equation it is clear that the N -th metric correction ${}_N R$ is associated to (or generated by) the "effective source" at the $N-1$ order ${}_{N-1} V$, linear in the energy momentum tensor. Whence, in the "full" Einstein's equation, also the Ricci tensor ${}_N R$ constructed on ${}_N g$ (or ${}_N R$) must be confronted with the Energy-momentum tensor at the order $N-1$. It then follows that the size that the components of the error-tensor must result to have, has to be consistent with the order of accuracy at which ${}_{N-1} T$ is known (for $N = 2$, see (3.8)).

Proof of theorem 2: In order to compute the components of the Ricci tensor from the metric (4.5), the formula

$$R_{ab} = \frac{1}{\sqrt{-g}} \left(\sqrt{-g} \{^m_{ab}\} \right)_{,m} - \left(\ln \sqrt{-g} \right)_{,ab} - \{^m_{la}\} \{^l_{mb}\}$$

is particularly convenient. A simple manipulation of the first element in the right hand side, shows that a term depending on the harmonic gauge can be extracted:

$$R_{ab} = \frac{1}{\sqrt{-g}} R^{mc}_{,m} [ab, c] + g^{mc} [ab, c]_{,m} - \\ - \left(\ln \sqrt{-g} \right)_{,ab} - \{^m_{la}\} \{^l_{mb}\}$$

The Christoffel symbols of the first and second kind

$$[ab, c] := \frac{1}{2} (\partial_a g_{bc} + \partial_b g_{ac} - \partial_c g_{ab})$$

$$\{^a_{bc}\} = g^{ad} [bc, d]$$

are given in Appendix B for the metric (4.5).

Using the expressions (4.4), (4.5) for g_{ab} , one has:

$${}_2 R_{00} = \frac{{}_2 R^{mc}_{,m}}{\sqrt{-{}_2 g}} [00, c]_2 - 4 \left(U_{,t} + W_{\alpha, \alpha} \right)_{,t} + \\ + \frac{g}{2} (1 + \pi) + g v^2 - g U + \frac{3p}{2} + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

On the other hand one has also:

$${}_1T_{00} - \frac{1}{2} {}_1g_{00} {}_1T = \frac{\rho}{2} (1+\pi) + \rho v^2 - \rho U + \frac{3P}{2} + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

Hence:

$${}_2E_{00} = - \frac{{}_2R^{mc}_{,m}}{\sqrt{-{}_2g}} [00, c]_2 - {}_2R^{0a}_{,ta} +$$

(4.9)

$$+ [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

Furthermore,

$${}_2R_{\alpha\beta} = - \frac{{}_2R^{mc}_{,m}}{\sqrt{-{}_2g}} [\alpha\beta, c]_2 + 2 \left[\left(W_{\alpha, t} + \frac{1}{8\pi} I_{-1, \gamma} [\alpha \gamma] \right)_{, \beta} + \right. \\ \left. + \left(W_{\beta, t} + \frac{1}{8\pi} I_{-1, \gamma} [\beta \gamma] \right)_{, \alpha} \right] + \\ + \delta_{\alpha\beta} \left[\frac{\rho}{2} (1+\pi) - \frac{P}{2} + \rho U \right] + \rho v_{\alpha} v_{\beta} + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

and

$${}_1T_{\alpha\beta} - \frac{1}{2} {}_1g_{\alpha\beta} {}_1T = \delta_{\alpha\beta} \left[\frac{\rho}{2} (1+\pi) - \frac{P}{2} + \rho U \right] + \rho v_{\alpha} v_{\beta} + \\ + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

Hence

$$\begin{aligned}
 {}_2E_{\alpha\beta} = & -\frac{{}_2R^{mc}_{,m}}{\sqrt{-{}_2g}} [\alpha\beta, c]_2 + \frac{1}{2} [{}_2R^{\alpha b}_{,b\beta} + {}_2R^{\beta b}_{,b\alpha}] + \\
 & + [\varepsilon^4\delta, \varepsilon^2\delta^3]
 \end{aligned} \tag{4.10}$$

Finally one obtains

$$\begin{aligned}
 {}_2R_{0\alpha} = & -\frac{{}_2R^{mc}_{,m}}{\sqrt{-{}_2g}} [0\alpha, c]_2 - 2 \left(U_{,t} + \frac{1}{8\pi} I_{-1,t} [00] + \right. \\
 & + W_{\beta,\beta} + \frac{1}{8\pi} I_{-1,\beta} [0\beta] - \frac{1}{16\pi} I_{0,\beta t} [\beta^{\parallel}] \Big)_{,\alpha} + \\
 & + 8U(U_{,t} + W_{\beta,\beta})_{,\alpha} + 4U_{,\alpha}(U_{,t} + W_{\beta,\beta}) - \\
 & - g(1 + 2U + v^2 + \pi + P/g) v_{\alpha} + 2g W_{\alpha} + \\
 & + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4]
 \end{aligned}$$

and

$$\begin{aligned}
 {}_1T_{0\alpha} - \frac{1}{2} g_{0\alpha} {}_1T = & -g(1 + v^2 + 2U + \frac{P}{g} + \pi) v_{\alpha} + \\
 & + 2g W_{\alpha} + [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4]
 \end{aligned}$$

Hence

$$\begin{aligned}
 {}_2E_{0\alpha} = & -\frac{{}_2R^{mc}_{,m}}{\sqrt{-{}_2g}} [0\alpha, c] - \frac{1}{2} {}_2R^{ob}_{,b\alpha} + \\
 & + 8U {}_2R^{ob}_{,\alpha b} + 4U_{,\alpha} {}_2R^{ob}_{,b} + \\
 & + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]
 \end{aligned} \tag{4.11}$$

Equations (4.9), (4.10) and (4.11) imply theorem 2.

(5): The relations between the approximate metrics and the approximate local equations of motion

The method described in the previous sections for the iterative solution of Einstein equation in the slow motion approximation is more efficient than needed: it provides g_{00} and $g_{\alpha\beta}$ at the same "level of accuracy", while the latter is needed at an accuracy less by a factor of order $[\varepsilon^2, \delta^2]$: in the metric element

$$ds^2 = dt^2 (g_{00} + 2 g_{0\alpha} v_\alpha + g_{\alpha\beta} v_\alpha v_\beta)$$

$g_{\alpha\beta}$ is multiplied by $v_\alpha v_\beta$, a quantity precisely of order δ^2 . Likewise for $g_{0\alpha}$ which appears multiplied by a factor of order δ . Looking at the result of the first iteration, one then sees that, in order to build up the metric element (3.14), instead of (3.13) one might also have considered

$$ds^2 = -(1-2U) dt^2 + dx^2 \quad (5.1)$$

The metric (5.1) suffices to obtain the equations of motion of the newtonian theory. It can therefore be called the "newtonian approximation" to the metric of General Relativity. Computing from (5.1) the Christoffel symbols, one obtains that the local equations of motion $T^{ab}_{;b} = 0$

$$\begin{aligned} \alpha=0: \quad \partial_t T^{00} + \partial_\alpha T^{0\alpha} + (\{^0_{00}\} + \{^b_{0b}\}) T^{00} + \\ + (2\{^0_{0\alpha}\} + \{^b_{\alpha b}\}) T^{0\alpha} + \{^0_{\alpha\beta}\} T^{\alpha\beta} = 0 \end{aligned} \quad (5.2)$$

$$\begin{aligned}
 a=\alpha : \quad \partial_t T^{0\alpha} + \partial_\beta T^{\alpha\beta} + \left\{ \begin{matrix} \alpha \\ 00 \end{matrix} \right\} T^{00} + 2 \left\{ \begin{matrix} \alpha \\ \beta 0 \end{matrix} \right\} T^{0\beta} + \\
 + \left\{ \begin{matrix} b \\ 0b \end{matrix} \right\} T^{00} + \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} T^{\beta\gamma} + \left\{ \begin{matrix} b \\ \beta b \end{matrix} \right\} T^{\alpha\beta} = 0
 \end{aligned} \quad (5.3)$$

have the forms

$$\partial_t \rho + \partial_\alpha (\rho v_\alpha) = [\varepsilon^4, \varepsilon^2 \delta^2] \quad (5.4)$$

$$\partial_t (\rho v_\alpha) + \partial_\beta (\rho v_\alpha v_\beta + p \delta_{\alpha\beta}) = \rho \partial_\alpha U + [\varepsilon^4, \varepsilon^2 \delta^2] \quad (5.5)$$

If the error terms in these equations are replaced by zero, the equations reduce to the continuity equation and Euler's equation of motion, respectively.

Adding to (5.1) terms of the next order, leads to the "first post-newtonian approximation" (*)

$$\begin{aligned}
 ds^2 = & - \left(1 - 2U - \frac{1}{4\pi} I_{-1} [00 + \mu\mu] + 6U^2 \right) dt^2 - \\
 & - 8W_\alpha dx^\alpha dt + (1 + 2U) d\underline{x}^2
 \end{aligned} \quad (5.6)$$

(*) The absence in ${}_2g_{00}$ of terms of order $[\varepsilon^2 \delta]$ means that "1/2 post newtonian approximation" (for which g_{00} would have elements odd in time) coincides with the newtonian one. Likewise the 1 1/2 post-newtonian approximation is found to coincide with the 1 P.N.A. Only ~~at~~ the 2 1/2 PNA one has terms in g odd in time, and it is precisely at this order that radiation reaction effects first make their appearance.

Using the Christoffel symbols appropriate for this metric (see appendix B), one obtains for (5.2), (5.3) respectively

$$\begin{aligned} & \partial_t \left[g (1 + v^2 + 2U + \pi) \right] + \partial_\alpha \left[g v_\alpha (1 + v^2 + 2U + p/g + \pi) \right] + \\ & + g U_{,t} = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \end{aligned} \quad (5.7)$$

$$\begin{aligned} & \partial_t \left[g v_\alpha (1 + v^2 + 2U + \pi + p/g) \right] + \partial_\beta \left[g v_\alpha v_\beta (1 + v^2 + 2U + \pi + p/g) + \right. \\ & + p(1 - 2U) \delta_{\alpha\beta} \left. \right] + g \left\{ - U_{,\alpha} + 4 U_{,\alpha}^2 - 4 W_{\alpha,t} - \frac{1}{8\pi} I_{-1,\alpha} [00 + \right. \\ & + \mu\mu] \left. \right\} - g U_{,\alpha} (v^2 + 2U + \pi) + 4 g v_\beta (W_{\beta,\alpha} - W_{\alpha,\beta}) + \\ & + 4 g v_\alpha U_{,t} + 4 U_{,\beta} (g v_\alpha v_\beta + p \delta_{\alpha\beta}) - \\ & - (g v^2 + 3p) U_{,\alpha} = \\ & = [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \end{aligned} \quad (5.8)$$

For the newtonian local equations, obtained from (5.4), (5.5) dropping the error terms, the name of equations of motion is appropriate since these equations determine $g(x,t)$, $v_\alpha(x,t)$ uniquely in terms of initial data $g(x,0)$, $v_\alpha(x,0)$.

For the post-newtonian equations, obtained from (5.7), (5.8) by dropping the error terms, this is not the case, for these equations do not give the time-derivatives of g and v_α in terms of their initial values.

But it is possible to transform (5.7), (5.8) into equations which, after dropping error terms, do in fact have the form of equations of motion in the newtonian sense. For this purpose one uses (5.4) and (5.5) and their consequences

$$U_{,t} + W_{\alpha,\alpha} = [\varepsilon^4, \varepsilon^2 \delta^2] \quad (5.9)$$

$$W_{\alpha,t} + \frac{1}{8\pi} I_{-1,\beta}[\alpha\beta] = [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (5.10)$$

in which the stress potential is given (recall (4.2)) by

$$I_{-1}[\alpha\beta] := \int [\rho v_\alpha v_\beta + p \delta_{\alpha\beta} + 2 U_{,\alpha} U_{,\beta} - \delta_{\alpha\beta} (\nabla U)^2]' |x-x'|^{-1} d^3 x'$$

in the post-newtonian equations (5.7), (5.8), in order to convert time derivatives of U and W_α (or g, g_α) into space derivatives (and vice-versa), provided that the size of the errors allows it. For instance in (5.8) the term $g v_\alpha U_{,t}$ can be converted into a spatial derivative modulo the error specified in the right hand side of (5.8). In fact, using (5.9)

$$\begin{aligned} g v_\alpha U_{,t} &= -g v_\alpha W_{\beta,\beta} + g v_\alpha [\varepsilon^4, \varepsilon^2 \delta^2] \\ &= -g v_\alpha W_{\beta,\beta} + [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \end{aligned}$$

Therefore, if the rules (3.8) are valid, one can insert (5.9), (5.10) into the post-newtonian terms occurring in (5.7), (5.8). The latter equations can therefore be rewritten in different ways, each expression being equivalent to (5.7), (5.8) modulo the lower order equations of motion (5.9), (5.10) and the "rules of the game" (3.8).

One can use this freedom to prove that (5.7), (5.8) can in particular be rewritten in such a way that only the density ρ and the velocity v_α respectively appear differentiated with respect to time.

A different formulation of the local equations of motion, still equivalent to (5.7), (5.8) modulo (3.8), will be derived in the next investigations, with the purpose of describing the motion of extended bodies.

Appendix A

Method of computing the components of the metric tensor g from the metric deviation k^{ab}

In sections 3, 4 we have seen how we can generate the higher order components of the "metric correction":

$$k^{ab} := -\sqrt{-g} g^{ab} + \eta^{ab} \quad (A1)$$

from the lower order ones, by iteration of the "slow motion" version of the reduced Einstein equations. The aim of this appendix is to answer the two following questions:

- a) how can we be sure that the metric generated at each time with the iteration method keeps the Lorentzian signature:

$$-g > 0 \quad (A2)$$

- b) how can we obtain g , g_{ab} and g^{ab} from k^{ab} ?

To answer question a), we consider the characteristic polynomial for a 4×4 matrix $A = A^a_b$

$$\begin{aligned} \det(\lambda \mathbb{1} - A) &= \lambda^4 - A_1 \lambda^3 + \frac{1}{2} (A_1^2 - A_2) \lambda^2 - \\ &- \left(\frac{1}{3} A_3 + \frac{1}{6} A_1^3 - \frac{1}{2} A_1 A_2 \right) \lambda + \left(\frac{1}{24} A_1^4 - \frac{1}{4} A_1^2 A_2 + \right. \\ &\left. + \frac{1}{3} A_1 A_3 + \frac{1}{8} A_2^2 - \frac{1}{4} A_4 \right) \end{aligned} \quad (A3)$$

$\mathbb{1}$ denotes the unit 4 x 4 matrix; we also use the notation

$$A_\nu := \text{Tr} (A^\nu) \quad (\nu=1, 2, 3, 4)$$

for the traces of the powers of A.

Setting $\lambda = 0$ in (A.3) we have:

$$\begin{aligned} \det A = & \frac{1}{24} A_1^4 - \frac{1}{4} A_1^2 A_2 + \frac{1}{3} A_1 A_3 \\ & + \frac{1}{8} A_2^2 - \frac{1}{4} A_4 \end{aligned} \quad (\text{A.4})$$

If $\det A \neq 0$, then substituting the matrix A for the variable λ in (A.3) we have:

$$\begin{aligned} A^{-1} = & \frac{-1}{\det A} \left[A^3 - A_1 A^2 + \frac{1}{2} (A_1^2 - A_2) A - \right. \\ & \left. - \left(\frac{1}{3} A_3 + \frac{1}{6} A_1^3 - \frac{1}{2} A_1 A_2 \right) \mathbb{1} \right] \end{aligned} \quad (\text{A.5})$$

Now, consider the case in which

$$A^a_b = S^a_b - R^a_b \quad (\text{A.6})$$

and define:

$$\eta^{ab} := A^{ab}$$

where:

$$A^{ab} := \eta^{bc} A^a_c \quad ; \quad \eta^{ab} := \sqrt{-g} g^{ab}$$

Inserting (A.6) into (A.4) we obtain:

$$\begin{aligned}
 -g = \det A = & 1 - R_1 + \frac{1}{2} R_1^2 - \frac{1}{2} R_2 - \frac{1}{6} R_1^3 + \frac{1}{2} R_1 R_2 - \\
 & - \frac{1}{3} R_3 + \frac{1}{24} R_1^4 - \frac{1}{4} R_1^2 R_2 + \\
 & + \frac{1}{3} R_1 R_3 + \frac{1}{8} R_2^2 - \frac{1}{4} R_4
 \end{aligned} \tag{A.7}$$

where, as above:

$$R_\nu := \text{Tr}(K^\nu) \quad (\nu = 1, 2, 3, 4)$$

The requirement of Lorentzian signature

$$-g > 0$$

is now imposed on the exact equation (A.7).

Assume $0 < \mu \ll 1$ and that

$$|R^{ab}| \leq \frac{\mu}{4} \tag{A.8}$$

holds for each component of k . Then

$$|R_\nu| \leq \mu^\nu \tag{A.8a}$$

Neglecting now in (A.7) the positive terms, we have as lower limit:

$$\begin{aligned}
 -g \geq & 1 - |R_1| - \frac{1}{2} |R_2| - \frac{1}{6} |R_1^3| - \frac{1}{2} |R_1 R_2| - \frac{1}{3} |R_3| - \\
 & - \frac{1}{4} R_1^2 |R_2| - \frac{1}{3} |R_1 R_3| - \frac{1}{4} |R_4|
 \end{aligned}$$

so that, for $|k_\nu| \leq \mu^\nu$, we have:

$$-g \geq 1 - \mu - \frac{\mu^2}{2} - \mu^3 - \frac{5}{6} \mu^4 \quad (\text{A.9})$$

The right hand side of (A.8) is positive provided

$$\mu \leq \frac{1}{2} \quad (\text{A.10})$$

We conclude that if the parameters ε^2, δ accounting for the typical sizes of the gravitational potential and of the velocities respectively, are so small that $|k^{ab}| \leq \frac{1}{8}$, then g_{ab} is Lorentzian.

We now consider problem b), that is the computation of g from the knowledge of k^{ab} .

To this purpose, define the three polynomials κ, Δ_3 and Δ_4 :

$$\Delta_4 := \frac{1}{24} k_1^4 - \frac{1}{4} k_1^2 k_2 + \frac{1}{3} k_1 k_3 + \frac{1}{8} k_2^2 - \frac{1}{4} k_4 \quad (\text{A.11})$$

$$\Delta_3 := -\frac{1}{6} k_1^3 + \frac{1}{2} k_1 k_2 - \frac{1}{3} k_3 \quad (\text{A.12})$$

$$\kappa := -k_1 + \frac{1}{2} k_1^2 - \frac{1}{2} k_2 + \Delta_3 + \Delta_4 \quad (\text{A.13})$$

One easily sees that:

$$\begin{aligned} |\Delta_4| &\leq \mu^4 \quad ; \quad |\Delta_3| \leq \mu^3 \\ |\kappa| &\leq \mu + \mu^2 + \mu^3 + \mu^4 \end{aligned} \quad (\text{A.14})$$

Consider now the following Taylor expansion ($-g = 1 + x$)

$$(-g)^{-1/2} = 1 - \frac{1}{2}x + \frac{3}{8}x^2 - \frac{5}{16}x^3 \left(1 - \frac{7}{8}x + \dots \right) \quad (\text{A.15})$$

where the dots denote all the remaining terms of the expansion. Since their coefficients are in modulo all less than one, introducing the notation

$$\Delta^{(3)} := \frac{5}{16}x^3 \left(1 - \frac{7}{8}x + \dots \right) \quad (\text{A.16})$$

one can write

$$\Delta^{(3)} \leq \frac{5}{16}x^3 \left(1 + |x| + |x|^2 + \dots \right) = \frac{5}{16} \frac{x^3}{1 - |x|}$$

Estimates (A.14) give:

$$|\Delta^{(3)}| \leq \mu^3 \frac{16875}{512} < 40 \mu^3 \quad (\text{A.17})$$

From (A.13), (A.15) and (A.16) we obtain:

$$(-g)^{-1/2} = 1 + \frac{1}{2}K_1 + \frac{1}{8}K_1^2 + \frac{1}{4}K_2 + K(3, 8) \quad (\text{A.18})$$

with the definitions:

$$K(3, 8) := -\frac{1}{2}\Delta_3 - \Delta^{(3)} - \frac{1}{2}\Delta_4 + \frac{3}{8}H(3, 8)$$

$$H(3, 8) := \left(-K_1 + \frac{1}{2}K_1^2 - \frac{1}{2}K_2 + \Delta_3 + \Delta_4 \right)^2 - K_1^2$$

For both K and H , which contain terms of third and higher order in k , an upper bound can be easily set up with the help of (A.8), (A.14):

$$|K(3,8)| \leq \frac{8113}{256} \mu^3 < 36 \mu^3 \quad (\text{A.19})$$

Hence, if in (A.18) one considers, as approximation, the terms of the first two orders in k , the resulting value of $(-g)^{-1/2}$ differs from the exact one by a quantity non exceeding (A.19):

$$\left| (-g)^{-1/2} - \left(1 + \frac{1}{2} R_1 + \frac{1}{8} R_1^2 + \frac{1}{4} R_2 \right) \right| < 36 \mu^3 \quad (\text{A.20})$$

Consider now:

$$g^{ab} = (\eta^{ab} - R^{ab}) (-g)^{-1/2} \quad (\text{A.21})$$

Also:

$$\begin{aligned} g^{ab} = & \left(1 + \frac{1}{2} R_1 + \frac{1}{8} R_1^2 + \frac{1}{4} R_2 \right) \eta^{ab} - \left(1 + \frac{1}{2} R_1 \right) R^{ab} + \\ & + K(3,8) \eta^{ab} - R^{ab} \left(\frac{1}{8} R_1^2 + \frac{1}{4} R_2 + K(3,8) \right) \end{aligned} \quad (\text{A.22})$$

is the exact expression of g^{ab} in terms of k^{ab} . Whence we can once again evaluate the error committed in taking only the terms up to the second order in k :

$$\begin{aligned}
 & \left| g^{ab} - \left[\left(1 + \frac{1}{2} R_1 + \frac{1}{8} R_1^2 + \frac{1}{4} R_2 \right) \eta^{ab} - \left(1 + \frac{1}{2} R_1 \right) K^{ab} \right] \right| \leq \\
 & \leq \frac{27531}{512} \mu^3 < 54 \mu^3
 \end{aligned} \tag{A.23}$$

So we can write

$$g^{ab} = \left(1 + \frac{1}{2} R_1 + \frac{1}{8} R_1^2 + \frac{1}{4} R_2 \right) \eta^{ab} - \left(1 + \frac{1}{2} R_1 \right) K^{ab} + K_{ab}^{(0)}(3,8) \tag{A.24}$$

with $K_{ab}^{(0)}(3,8) \leq 54 \mu^3$

The next task is to work out a formula analogous to (A.22) for the covariant components of the metric tensor.

To this purpose, following Peres [12], define:

$$h^{ab} := (-g)^{1/4} g^{ab} \tag{A.25}$$

so that:

$$\eta^{ab} - K^{ab} = (-g)^{1/4} h^{ab}$$

In [12] it is pointed out that:

$$\begin{aligned}
 h_{ab} = & -\eta_{am} h^{mn} \eta_{np} h^{pq} \eta_{qr} h^{rs} \eta_{sb} + h_1 \eta_{am} h^{mn} \eta_{np} h^{pq} \eta_{qb} + \\
 & + \frac{1}{2} [h_2 - (h_1)^2] \eta_{am} h^{mn} \eta_{nb} + \eta_{ab} \left[\frac{1}{6} (h_1)^3 + \right. \\
 & \left. + \frac{1}{3} h_3 - \frac{1}{2} h_1 h_2 \right]
 \end{aligned} \tag{A.26}$$

where, as usual,

$$h_\nu := \text{Tr}(h^\nu) \quad (\nu = 1, 2, 3, 4)$$

From (A.25), (A.26) we get:

$$\begin{aligned} (-g) \, g^{ab} &= (-g)^{3/4} h_{ab} = \\ &= \left(1 - R_1 + \frac{1}{2} R_1^2 - \frac{1}{2} R_2 + \frac{1}{2} R_1 R_2 - \frac{1}{6} R_1^3 - \frac{1}{3} R_3\right) \eta_{ab} + \\ &+ \left(1 - R_1 + \frac{1}{2} R_1^2 - \frac{1}{2} R_2\right) \eta_{ae} \eta_{bm} R^{em} + \\ &+ (1 - R_1) \eta_{ae} \eta_{cm} \eta_{bs} R^{em} R^{cs} + \\ &+ \eta_{ae} \eta_{cm} \eta_{dn} \eta_{bp} R^{em} R^{cd} R^{np} \end{aligned} \quad (\text{A.27})$$

Since on the other hand:

$$g_{ab} = \sqrt{-g} \, g^{ab}$$

one sees that the product of $(-g)^{-1/2}$ with the right hand side of (A.27) gives precisely g_{ab} in an exact form:

$$\begin{aligned} g_{ab} &= \left(1 - \frac{1}{2} R_1 + \frac{1}{8} R_1^2 - \frac{1}{4} R_2\right) \eta_{ab} + \left(1 - \frac{1}{2} R_1\right) \eta_{ae} \eta_{bm} R^{em} \\ &+ \eta_{ae} \eta_{cm} \eta_{bs} R^{em} R^{cs} + K_{ab}^{(1)}(3,8) \end{aligned} \quad (\text{A.28})$$

where $K_{ab}^{(1)}(3,8)$ stays for the remaining terms: writing down explicitly all of them and using (A.8), (A.8a), we have found:

$$|K_{ab}^{(1)}(3, 8)| < 111 \mu^3 \quad (\text{A.29})$$

which is the maximum error committed in keeping terms up to the second order in (A.28) for the calculation of g_{ab} .

We notice that the error bound in (A.29) is larger than that in (A.23) which is in turn larger than that of (A.19). This behaviour is precisely what one expects since at each step in the computation

$$(-g)^{-1/2} \rightarrow g^{ab} \rightarrow g_{ab}$$

in terms of k^{ab} , we have added terms of third and higher order in k , hence modifying the entity of the error.

The preceding equations and error estimates are exact. They improve similar equations with formal " $O(\frac{1}{c^n})$ " error-terms given, e.g., in appendix A in [2].

Appendix B

Christoffel symbols of the metric (4.3)

$$\left\{ \begin{matrix} 0 \\ 00 \end{matrix} \right\} = -U_{,t} + 2 U_{,t}^2 - \frac{1}{8\pi} I_{-1,t} [00 + \mu\mu] + 4 W_{\alpha} U_{,\alpha}$$

$$\left\{ \begin{matrix} \alpha \\ 00 \end{matrix} \right\} = -U_{,\alpha} + 4 U_{,\alpha}^2 - 4 W_{\alpha,t} - \frac{1}{8\pi} I_{-1,\alpha} [00 + \mu\mu]$$

$$\left\{ \begin{matrix} 0 \\ \alpha 0 \end{matrix} \right\} = -U_{,\alpha} + 2 U_{,\alpha}^2 - \frac{1}{8\pi} I_{-1,\alpha} [00 + \mu\mu]$$

$$\begin{aligned} \left\{ \begin{matrix} 0 \\ \alpha \beta \end{matrix} \right\} = & 2(W_{\alpha,\beta} + W_{\beta,\alpha}) + \delta_{\alpha\beta} U_{,t} + \frac{1}{4\pi} (I_{-1,\alpha} [0\beta] + \\ & + I_{-1,\beta} [0\alpha] + I_{-1,t} [\alpha\beta]) - \frac{1}{8\pi} (I_{0,\alpha t} [\beta||] + \\ & + I_{0,\beta t} [\alpha||] - \delta_{\alpha\beta} I_{-1,t} [00 - \mu\mu]) - \\ & - 8(W_{\alpha} U_{,\beta} + W_{\beta} U_{,\alpha}) + 4\delta_{\alpha\beta} W_{\gamma} U_{,\gamma} \end{aligned}$$

$$\begin{aligned} \left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} = & \delta_{\alpha\beta} U_{,\gamma} + \delta_{\alpha\gamma} U_{,\beta} - \delta_{\beta\gamma} U_{,\alpha} - 2(\delta_{\alpha\beta} U_{,\gamma}^2 + \\ & + \delta_{\alpha\gamma} U_{,\beta}^2 - \delta_{\beta\gamma} U_{,\alpha}^2) + \frac{1}{8\pi} (\delta_{\alpha\beta} I_{-1,\gamma} [00 - \mu\mu] + \\ & + \delta_{\alpha\gamma} I_{-1,\beta} [00 - \mu\mu] - \delta_{\beta\gamma} I_{-1,\alpha} [00 - \mu\mu]) + \\ & + \frac{1}{4\pi} (I_{-1,\beta} [\alpha\gamma] + I_{-1,\gamma} [\alpha\beta] - I_{-1,\alpha} [\beta\gamma]) \end{aligned}$$

$$\begin{aligned} \left\{ \begin{matrix} \alpha \\ \beta 0 \end{matrix} \right\} = & 2(W_{\beta,\alpha} - W_{\alpha,\beta}) + \delta_{\alpha\beta} U_{,t} + 8U(W_{\alpha,\beta} - W_{\beta,\alpha}) - \\ & - 2U_{,t}^2 \delta_{\alpha\beta} + \frac{1}{4\pi} (I_{-1,\alpha} [0\beta] - I_{-1,\beta} [0\alpha] + I_{-1,t} [\alpha\beta]) + \\ & + \frac{\delta_{\alpha\beta}}{8\pi} I_{-1,t} [00 - \mu\mu] - 4U_{,\alpha} W_{\beta} + \\ & + \frac{1}{8\pi} (I_{0,\beta t} [\alpha||] - I_{0,\alpha t} [\beta||]) \end{aligned}$$

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Chapter II

On the Definition of the Mass in the
Post-Newtonian Approximation

Abstract: A definition of the mass of a single body in an isolated, self gravitating N-body system, and of the mass of the whole system is investigated in the framework of the first post-newtonian approximation to General Relativity. For the mass of a single body, which depends on the sum of the masses of the baryons and on their gravitational energy of mutual interaction as well, it is found under which conditions it is constant in time. The mass of the whole system is compared with other definitions (Tolman's, Dixon's) already available in the literature, in the special case of a static, perfect fluid configuration.

(1): Introduction

The purpose of this chapter is to discuss the concept of mass of a body and of a system of bodies, in the framework of the post-newtonian equations of motion for a perfect fluid, derived in the previous chapter.

In newtonian theory the concept of mass is well defined independently of any field acting upon the physical system: if ϱ is the mass-density, and $\underline{v}$ the velocity-field, then the mass is defined by

$$M_{nw} = \int \varrho \, d^3x$$

and its equation of motion is

$$\frac{dM_{nw}}{dt} = 0 \quad (1.1)$$

as a consequence of the local equation of motion (continuity equation)

$$\partial_t \varrho + \partial_\mu (\varrho v_\mu) = 0 \quad (1.2)$$

independent of any field equation.

In General Relativity, Einstein's equation implies the local equations of motion because of the contracted Bianchi identities

$$\left. \begin{array}{l} G^{ab} = T^{ab} \\ G^{ab}_{;b} = 0 \end{array} \right\} \rightarrow T^{ab}_{;b} = 0. \quad (1.3)$$

In (1.3) the semicolon means covariant derivative with respect to a metric and, in general, non integrable connection on the space-time manifold. Whence (1.3) does not express any conservation law, and furthermore it depends on the field equations.

Thanks to the work of Landau and Lifshitz, another version of the local equations of motion is available

$$\left[(-g) (T^{ab} + t^{ab}) \right]_{,b} = 0, \quad (1.4)$$

where t^{ab} is the Landau-Lifshitz complex of the gravitational field.

Since in the Landau-Lifshitz version of the local equations of motion the ordinary divergence (that is the covariant derivative with respect to an auxiliary, flat connection) comes into play, it is straightforward to define from (1.4) conserved quantities, in the same way as in special relativity. In order to define the mass of a system of bodies, in section 2 the attention will be concentrated on the first integral

$$M = \int (-g) (T^{00} + t^{00}) d^3x \quad (1.5)$$

referred to as the A.D.M. mass, that is the total mass of an isolated system, considering the explicit expression for it with respect to the metric in the post-newtonian

approximation to General Relativity, derived in the previous chapter. We will then consider (sect. 3) a plausible definition for the mass of a body belonging to an N-body system. In sect. 4 we will look at the time-evolution of this mass according to (1.3) or (1.4), finding under which conditions this post-newtonian mass of a body of an N-body system is constant in time. The expression of the total mass (1.5) derived in a quasi-stationary, slow-motion, weak-field-approximation to an unknown, exact solution of the field equations, will be, in sect. 5, compared with the exact Tolman formula for the A.D.M. mass, valid for a static, spherically symmetric, perfect fluid solution of the Einstein equation. In sect. 6 the comparison is instead made with the mass of a perfect fluid, static, spherically symmetric, extended body, as defined in the framework of the Dixon's theory.

(2): The definition of the mass of an isolated system

A fundamental law of physics is the conservation of the baryonic number. Macroscopically, it is expressed covariantly in the form

$$(\varrho u^a)_{;a} = 0 \quad (2.1)$$

where u^a is the 4-velocity of the matter, and ϱ is the baryon density. An expression equivalent to (2.1) is

$$(\sqrt{-g} \varrho u^a)_{,a} = 0, \quad (2.2)$$

g denoting the determinant of the metric.

Eq. (2.2), defining the law of time evolution of the baryon density, has to be considered as an equation of motion, as (1.3). The final equation to be considered comes from the thermodynamical law

$$\varrho d\pi = \frac{p}{\varrho} d\varrho, \quad (2.3)$$

π being the internal specific energy, which, using (1.2), can be shown to imply

$$\varrho \frac{d\pi}{dt} = -p \partial_\mu v^\mu + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]. \quad (2.4)$$

The basic equations we will need in the following are then:

$$\left[(-g) (T^{0a} + t^{0a}) \right]_{,a} = 0 \quad (2.5)$$

$$(\rho u^\alpha \sqrt{-g})_{,t} + (\rho u^\mu \sqrt{-g})_{,\mu} = 0 \quad (2.6)$$

$$\rho \frac{d\pi}{dt} = -\rho \partial_\mu v_\mu + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (2.7)$$

In the newtonian approximation there is no difference between eqs. (2.5) and (2.6): they both reduce to (1.2) and its integration over the support of the density gives the conservation law for the associated mass. Consider now the components of the metric tensor g_{ab} in the post-newtonian approximation, in the case in which the material composing the isolated system is described by a perfect-fluid energy-momentum tensor (see ch.1, sect.5). Equation (2.5) can then be written down explicitly:

$$\begin{aligned} & \partial_t [\rho (1 + v^2 + 6U + \pi) - 7(\nabla U)^2] + \\ & + \partial_\mu [\rho v_\mu (1 + v^2 + 6U + \pi + p/\rho)] + \\ & + 2\partial_\mu [3U_{,t} U_{,\mu} + 4 U_{,\nu} (W_{\nu,\mu} - W_{\mu,\nu})] = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \end{aligned} \quad (2.8)$$

where the right hand side denotes the order of magnitude of the terms which are neglected in the first post-newtonian approximation.

The vector potential is defined by:

$$8\pi W_\alpha(x, t) := \int \rho(x', t) v_\alpha(x', t) |x - x'|^{-1} d^3x \quad (2.9)$$

and satisfies

$$\nabla^2 W_\alpha = -\frac{1}{2} \rho v_\alpha \quad (2.10)$$

The third bracketed term in (2.8) has non compact support, but it falls-off as r^{-5} , going away from $\text{supp}(\varrho)$.

Integration of (2.8) over a sphere of radius r surrounding the system, provides, by means of the rules (3.8) in ch.1

$$\begin{aligned} \frac{d}{dt} \int [\varrho(1+v^2+6U+\pi) - 7(\nabla U)^2] d^3x = \\ = [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] + O(r^{-2}), \end{aligned} \quad (2.11)$$

or also, using the identity

$$(\nabla U)^2 = \frac{1}{2} \varrho U + \frac{1}{2} \nabla \cdot (\nabla U^2),$$

we have from (2.11)

$$\frac{dM}{dt} = [\varepsilon^6, \varepsilon^4\delta^2, \varepsilon^2\delta^4] + O(r^{-2}) \quad (2.12)$$

with M defined by

$$M := \int \varrho \left(1 + v^2 + \frac{5}{2} U + \pi \right) d^3x. \quad (2.13)$$

On the other hand the law of conservation of the baryon number (2.6) gives

$$\frac{dM_0}{dt} = 0 \quad (2.14)$$

with M_0 defined by

$$M_0 := \int \varrho u^0 \sqrt{-g} d^3x = \int \varrho \left(1 + \frac{1}{2} v^2 + 3U \right) d^3x + [\varepsilon^4\delta, \varepsilon^2\delta^3] \quad (2.15)$$

Subtraction of (2.14) from (2.12) finally gives

$$\frac{dE}{dt} = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] + O(r^{-2}) \quad (2.16)$$

with E defined by

$$E := M - M_0 = \int \left(\frac{v^2}{2} - \frac{U}{2} + \pi \right) d^3x. \quad (2.17)$$

We note that in (2.12) or (2.16) the term falling off as r^{-2} is denoted by $O(r^{-2})$ and arising from the integration of the divergence term in (2.8), gives a contribution arbitrarily small, since we have the freedom of choosing the surface of integration at arbitrarily large distance from the material source. This surface needs not to be at infinity. In fact from (2.16) it follows that it is sufficient that the quantity $O(r^{-2})$ is as small as $[\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]$. The terms in (2.8) with non compact support having the form of a divergence are of order $[\varepsilon^4 \delta, \varepsilon^2 \delta^3]$. Whence $O(r^{-2})$ is roughly

$$[\varepsilon^4 \delta, \varepsilon^2 \delta^3] r^{-2}$$

It follows that if r satisfies

$$[\varepsilon^4 \delta, \varepsilon^2 \delta^3] r^{-2} = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4],$$

then the two terms in the right hand side of (2.12), or (2.16), are of the same order of magnitude.

The equations of motion (2.12), (2.14), and (2.16) are interpreted as the conservation law of the inertial mass, of the rest mass, and of the energy respectively, for the isolated system under consideration.

(3): The definition of the mass of a body in a N-body-system

Focusing attention on a body (A, say) belonging to an isolated N-body system (A,B,C,...), the definitions of its total mass, rest mass and energy are motivated by the following considerations:

a) with respect to the inertial, harmonic coordinate system, with origin, say, at the newtonian center of mass of the whole system, the body A will be seen in accelerated motion, so that the quantity (see (2.17))

$$\frac{1}{2} \int v^2(x,t) \quad x \in \text{supp}(A)$$

does not represent the density of kinetic energy coming only from the internal motions within the body A, but also from the overall motion of the body with respect to the origin of the coordinates.

b) likewise the gravitational energy density, proportional to

$$\int (x,t) U(x,t) \quad x \in \text{supp}(A)$$

represents the sum of the energy coming from the self-gravitation of the body, and from the gravitational interaction of the body itself with the other bodies of the system.

c) an individual body has a conserved rest-mass defined by (2.15) (see ^{also}eq. (2.14)), but not a conserved energy or total mass: in fact e.g. tidal interaction with the other bodies will determine its time evolution.

From a) and b) we see that the definition of the mass of a generic body A requires a decomposition of the velocity field $\underline{v}$ in an overall motion " $\dot{\underline{a}}(t)$ " (e.g. the motion of the center of mass) and an internal motion " $\dot{\underline{\xi}}$ " : e.g. for the body A

$$\underline{v}_{\alpha}(x,t) = \dot{\underline{a}}_{\alpha}(t) + \dot{\underline{\xi}}_{\alpha}(x,t) \quad x \in \text{supp}(A) \quad (3.1)$$

and a decomposition of the potentials into a part coming from the body A itself and a part coming from the interaction with the other bodies:

$$\begin{aligned} U(x,t) &= u_a(x,t) + U^{(a)}(x,t) \\ W_{\alpha}(x,t) &= w_{\alpha}^{(a)}(x,t) + W_{\alpha}^{(a)}(x,t) \end{aligned} \quad x \in \text{supp}(A) \quad (3.2)$$

where

$$8\pi u_a(x,t) := \int_A \rho' |x-x'|^{-1} d^3x'; \quad 8\pi w_{\alpha}^{(a)}(x,t) := \int_A \rho' v'_{\alpha} |x-x'|^{-1} d^3x'$$

are the self potentials, and

$$8\pi U^{(a)}(x,t) := \sum_{B \neq A} \int_B \rho' |x-x'|^{-1} d^3x'; \quad 8\pi W_{\alpha}^{(a)}(x,t) := \sum_{B \neq A} \int_B \rho' v'_{\alpha} |x-x'|^{-1} d^3x'$$

are the external potentials.

Furthermore, we need the introduction of a local (say orthogonal) coordinate system attached to the representative point of the body, in order to define quantities like the self-energies in the local comoving frame of the body. Hence we specify the representative point of each body A, B, C, ... by requiring it to satisfy the center-of-mass condition at the newtonian approximation: e.g. for the body A, (a_1, a_2, a_3) are the coordinates of the center of mass, defined by:

$$M_A^{(nw)} a_\alpha := \int_A \rho x_\alpha d^3x \quad (3.3)$$

where $M_A^{(nw)}$ is the mass defined according to the newtonian approximation

$$M_A^{(nw)} := \int_A \rho d^3x .$$

We then assume that there exists a local coordinate system $(\tilde{x}, \tilde{t})$ with origin at "a" : in these coordinates we have

$$\dot{\tilde{a}}_\alpha = 0$$

and the (approximate) law of transformation of the velocity and of the potentials from the global inertial frame to the local, comoving frame is, at the first order,

$$\dot{\xi}_\alpha = \tilde{v}_\alpha + [\delta^1]; \quad U = \tilde{U} + [\varepsilon^2 \delta] \quad (3.4)$$

that is, a Galilei transformation which leaves relative velocity and potential invariant.

It is then possible to define (in the twiddled coordinates) the energy of the body A as follows:

$$\tilde{E}_A := \int_A \tilde{\rho} \left(\frac{\tilde{v}^2}{2} - \frac{\tilde{u}}{2} + \tilde{\pi} \right) d^3x \quad (3.5)$$

The representation of this quantity in the global, inertial coordinate frame, at rest with the center of mass of the whole system, is obtained using (3.4):

$$\tilde{E}_A = \int_A \rho \left(\frac{1}{2} \dot{\xi}^2 - \frac{u}{2} + \pi \right) d^3x + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

Defining

$$E_A = \int_A \varrho \left(\frac{1}{2} \dot{\xi}^2 - \frac{u}{2} + \Pi \right) d^3x, \quad (3.6)$$

we add this quantity to the rest mass of the body (see (2.15)):

$$M_A^O = \int_A \varrho \left(1 + \frac{v^2}{2} + 3U \right) d^3x. \quad (3.7)$$

We then define, following Spyrou ([10], eq. (3.22)), the inertial mass of the body A as the sum of its rest mass and of its self-energy, in the global, inertial frame:

$$\begin{aligned} M_A &:= M_A^O + E_A = \\ &= \int_A \varrho^* \left(1 + \frac{1}{2} \dot{\xi}^2 - \frac{u}{2} + \Pi \right) d^3x, \end{aligned} \quad (3.8)$$

with

$$\varrho^* := \varrho u^O \sqrt{-g} = \varrho \left(1 + \frac{v^2}{2} + 3U \right) + [\varepsilon^4 \delta, \varepsilon^2 \delta^3].$$

(4): The equation of motion for the inertial mass

In this section we will establish the law of time-evolution for the inertial mass defined in (3.8) using the general equations of motion (2.5), (2.6) in the post-newtonian approximation.

Indeed the only part of the inertial mass which undergoes to non trivial time-evolution is the energy (3.6), since the rest mass M_A^0 is independent of time, because of the conservation of the baryonic number (2.1).

Also, in order to compute the value of the time-derivative of E_A , we will follow this strategy: subtract (2.6) from (2.5); Use the equations of motion at the newtonian approximation, the decompositions (3.1) and (3.2) for the velocity and potential, and the newtonian center-of-mass condition (3.3). Omitting the straightforward algebra, from the comparison of (2.5) and (2.6), and integration over the volume of the generic body A, we have

$$\begin{aligned} & \frac{d}{dt} \int_A \rho \left(\frac{V^2}{2} + 3U + \pi \right) d^3x - 7 \frac{d}{dt} \int_A (\nabla U)^2 d^3x + \\ & + 2 \int_A \left[3U_{,t} U_{,\alpha} + 4U_{,\beta} (W_{\beta,\alpha} - W_{\alpha,\beta}) \right] d^3x = [\varepsilon^4, \varepsilon^4 S^2, \varepsilon^2 S^4] \quad (4.1) \end{aligned}$$

Next we use the decomposition of the potentials (3.2) and the relations

$$\begin{aligned} U_{,t} + W_{\alpha\alpha} &= [\varepsilon^4, \varepsilon^2 S^2] \\ \frac{d}{dt} \int_A \rho U d^3x &= \int_A \rho \frac{dU}{dt} d^3x + [\varepsilon^4 S^2, \varepsilon^2 S^4] \end{aligned}$$

(which are consequences of the newtonian continuity equation (1.2)), in order to bring (4.1) into the form

$$\begin{aligned} \frac{d}{dt} \int_A \rho \left[\frac{v^2}{2} - \frac{1}{2} u_a + \pi \right] d^3x &= \\ &= \int_A \rho v_\alpha \partial_\alpha U^{(a)} d^3x + [\varepsilon^6, \varepsilon^4 S^2, \varepsilon^2 S^4]. \end{aligned} \quad (4.2)$$

Finally we use the decomposition (3.1) for the velocity, and (3.3), in order to obtain from (4.2)

$$\frac{dE_A}{dt} = \int_A \rho \dot{\xi}_\alpha \partial_\alpha U^{(a)} d^3x + [\varepsilon^6, \varepsilon^4 S^2, \varepsilon^2 S^4] \quad (4.3)$$

which is the desired equation of motion for the self-energy of the body A.

We notice from (4.3) that the time evolution of E_A is determined by the action of the external potential $U^{(a)}$ on A, while the self force makes no work, besides giving no contribution to the acceleration.

For astronomical applications of the gravitational N-body problem, it is often assumed that the typical size of the bodies (here taken as unit length) is small compared with their mutual separation, so that it is meaningful to consider an expansion of the external potential at a point of the body under consideration, in negative powers of the distance from the other bodies. This means that in the equations of motion one describes the time-evolution of a certain property of the body via the coupling of its multipole moments to the external field. This idea was elegantly developed by Dixon [1] and we want to use it in order to write (4.3) in a different form.

According to investigations on close binary systems (see e.g. [5] pg.27), one can show that to a good accuracy the tidal action of one component of the system upon another can be considered as being due to a mass point, i.e. we can neglect the multipole structure of the bodies B,C,... and its effect on the field at a point of the body A. Hence we can consider the expansion ($x \in \text{supp}(A)$)

$$\begin{aligned}
 8\pi \int_A \dot{\xi}_\alpha U^{(\alpha)}(x,t) &= - \sum_{B \neq A} M_B \frac{(x_\alpha - b_\alpha)}{|x-b|^3} + [\varepsilon^2 \delta] \varepsilon \\
 &= - \sum_{B \neq A} \frac{M_B}{|a-b|^3} \left\{ (a_\alpha - b_\alpha) + (x_\beta - a_\beta) \left[\delta_{\alpha\beta} - \frac{3(a_\alpha - b_\alpha)(a_\beta - b_\beta)}{|a-b|^2} \right] - \right. \\
 &\quad \left. - \frac{3}{2} \frac{(x_\beta - a_\beta)(x_\gamma - a_\gamma)}{|a-b|^2} \left[3\delta_{\alpha\gamma}(a_\beta - b_\beta) - \frac{5(a_\alpha - b_\alpha)(a_\beta - b_\beta)(a_\gamma - b_\gamma)}{|a-b|^2} \right] \right\} \\
 &\quad + O(|a-b|^{-5}) + [\varepsilon^2 \delta]
 \end{aligned} \tag{4.4}$$

where "b" denotes the center-of-mass location of the body B and round brackets denote complete symmetrization. Inserting (4.4) into (4.3), and taking into account the center-of-mass condition, we rewrite (4.3) in the form

$$\begin{aligned}
 8\pi \int_A \dot{\xi}_\alpha U^{(\alpha)} d^3x &= - P_{(\alpha\beta)} \sum_{B \neq A} \frac{M_B}{|a-b|^3} \left(\delta_{\alpha\beta} - 3 \frac{(a_\alpha - b_\alpha)(a_\beta - b_\beta)}{|a-b|^2} \right) + \\
 &+ P_{(\alpha\beta\gamma)} \frac{3}{2} \sum_{B \neq A} \frac{M_B}{|a-b|^4} \left(3\delta_{\alpha\beta} \frac{a_\gamma - b_\gamma}{|a-b|} - 5 \frac{(a_\alpha - b_\alpha)(a_\beta - b_\beta)(a_\gamma - b_\gamma)}{|a-b|^3} \right) + \\
 &+ O(|a-b|^{-5}) + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]
 \end{aligned} \tag{4.5}$$

where, following Dixon, we have introduced the quantities

$$P_{\alpha\beta_1 \dots \beta_m}(\mathbb{E}) := \int_A \dot{\xi}_\alpha \xi_{\beta_1} \dots \xi_{\beta_m} d^3x$$

We notice that in (4.5) only the symmetric and traceless parts of the P's come into play, and that, as Dixon points out, the values of the symmetric parts of the P's are determined in terms of the multipole moments of the mass as follows: ($n \geq 2$)

$$P_{(\alpha_1 \dots \alpha_n)} = \frac{1}{n} \frac{d}{dt} m_{\alpha_1 \dots \alpha_n} + [\varepsilon^4, \varepsilon^2 \delta^2] \quad (4.6)$$

with the mass-multipole moments defined by

$$m_{\alpha_1 \dots \alpha_n} := \int_A \rho \xi_{\alpha_1} \dots \xi_{\alpha_n} d^3 x$$

Inserting (4.6) into (4.5) and recalling that the rest mass M_A^0 is constant, we can finally write

$$\begin{aligned} 8\pi \frac{dM_A}{dt} = & -\frac{1}{2} \left(\frac{d}{dt} m_{\alpha\beta} \right) \sum_{B \neq A} \frac{M_B}{|a-b|^3} \left(\delta_{\alpha\beta} - \frac{3(a_\alpha - b_\alpha)(a_\beta - b_\beta)}{|a-b|^2} \right) + \\ & + \frac{1}{2} \left(\frac{d}{dt} m_{\alpha\beta\gamma} \right) \sum_{B \neq A} \frac{M_B}{|a-b|^4} \left[3 \delta_{(\alpha\beta} \frac{(a_{\gamma)} - b_{\gamma})}{|a-b|} - \right. \\ & \left. - \frac{5(a_\alpha - b_\alpha)(a_\beta - b_\beta)(a_\gamma - b_\gamma)}{|a-b|^3} \right] + \\ & + O(|a-b|^{-5}) + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (4.7) \end{aligned}$$

All the remaining infinitely many multipole moments of higher order are symbolized by $O(|a-b|^{-5})$, so that (4.7) is exact. Equation (4.7) shows that the time-evolution of the inertial mass of a body belonging to a N-body system is completely determined by the time-evolution of the traceless (in the sense of [7] pg.283) part of the infinitely many multipole moments associated to the mass-density.

If the body is in rigid rotation, then

$$\dot{\xi}_\alpha = \omega_{\alpha\beta}(t) \xi_\beta$$

with $\omega_{\alpha\beta} = \omega_{[\alpha\beta]}$. The equations of motion for the mass-moments take then the form

$$\frac{d}{dt} m_{\alpha_1 \dots \alpha_n} = n \omega_\beta (\alpha_1, m_{\alpha_2 \dots \alpha_n}) \quad (n \geq 2).$$

Consider first the mass-quadrupole moment ($n = 2$): referring its components to the principal axes of inertia of the body, and defining ω_α and $\hat{R}$ by

$$\frac{1}{2} \varepsilon_{\alpha\beta\gamma} \omega_\beta \gamma = \omega_\alpha$$

and

$$\hat{R}_\alpha = (a_\alpha - b_\alpha) / |a - b|$$

respectively, it easily verified that

$$\begin{aligned} -\frac{1}{6} \frac{d}{dt} m_{\alpha\beta} (\delta_{\alpha\beta} - 3\hat{R}_\alpha \hat{R}_\beta) &= \hat{R}_1 \hat{R}_2 \omega_3 (m_{11} - m_{22}) + \\ &+ \hat{R}_1 \hat{R}_3 \omega_2 (m_{33} - m_{11}) + \hat{R}_2 \hat{R}_3 \omega_1 (m_{22} - m_{33}). \end{aligned}$$

If the body is a spheroid, then $m_{11} = m_{22} \neq m_{33}$, say, and

$$\begin{aligned} -\frac{1}{6} \frac{d}{dt} m_{\alpha\beta} (\delta_{\alpha\beta} - 3\hat{R}_\alpha \hat{R}_\beta) &= \\ &= (m_{11} - m_{33}) (R_2 \omega_1 - R_1 \omega_2). \end{aligned}$$

From this equation it follows that, for arbitrary orientation of the angular-velocity vector,

$$\left(\frac{d}{dt} m_{\alpha\beta}\right) \sum_{B \neq A} \frac{M_B}{|a-b|^3} (\delta_{\alpha\beta} - 3\hat{R}_\alpha \hat{R}_\beta) = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]$$

if

$$\frac{m_{11} - m_{33}}{R^3} = [\varepsilon^2 \delta] \quad (4.8)$$

Eq. (4.8) is satisfied either by an almost spherical body, or for large mutual distances between pairs of bodies. Analogous considerations hold for the higher-order moments, taking into account that for spheroids it is

$$m_{\alpha_1 \dots \alpha_{2m+1}} = 0 \quad (m \geq 0)$$

if the moments are referred to the center of the bodies. Also if a body is in rigid rotation and satisfies (4.8), then (4.7) implies

$$\dot{M}_A = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (4.9)$$

Eq. (4.9) will be used in chapters 3 and 4 in the analysis of the motion of extended, perfect fluid and rigidly rotating spheroids, in the post-newtonian approximation.

(5): Comparison between the inertial and the Tolman's mass

In section 2 we have discussed a definition of the inertial mass which, within the post-newtonian framework, appears as the most plausible definition for the total mass of an isolated, self-gravitating system. The post-newtonian framework is however an approximation scheme to an unknown, and maybe non existing exact solution of the Einstein field equation. It is therefore worthwhile to compare the definition introduced in the approximate metric, with analog definitions valid in an exact framework. To this purpose we can use the form of an exact solution, which describes the gravitational field of a static, spherically symmetric and isolated body made of perfect fluid. The metric in this case is:

$$ds^2 = - e^{-2U} dt^2 + e^{2\Lambda} dr^2 + r^2 d\Omega$$

($-e^{-2U}$ being the norm of the time-like and hypersurface orthogonal Killing vector, Λ a function of r and $d\Omega$ the solid angle), and the mass is defined as (see [3] eq.2-3.17):

$$M_T = \int (\mu + 3p) \sqrt{-g} d^3x \quad (5.1)$$

μ being the mass-energy density, eigenvalue of the energy-momentum tensor. In this framework we can then consider a static, weak field approximation, in which

$$\mu = \rho(1 + \pi) ; \quad \sqrt{-g} = 1 + 2U + [\mathcal{E}^2 \delta]$$

Using isotropic coordinates, we have

$$M_T = \int [\varrho(1 + \pi) + 2U + 3p] d^3x + [\epsilon^4 \delta, \epsilon^2 \delta^3]$$

Eliminating the integrated pressure via the virial theorem:

$$\frac{1}{2} \int \varrho U d^3x = 3 \int p d^3x + [\epsilon^4 \delta, \epsilon^2 \delta^3] \quad (5.2)$$

we finally obtain:

$$M_T = \int \varrho \left(1 + \pi + \frac{5}{2} U \right) d^3x + [\epsilon^4 \delta, \epsilon^2 \delta^3] \quad (5.3)$$

which has to be compared with (2.13) (see below), derived within a quasi-stationary, weak-field, slow motion approximation, with no Killing vector and where a velocity field is present:

$$M = \int \varrho \left(1 + \pi + v^2 + \frac{5}{2} U \right) d^3x + [\epsilon^4 \delta, \epsilon^2 \delta^3] \quad (2.13)$$

We see that the diversity between the inertial mass (2.13) and the Tolman mass consists only in the velocity term, which in the latter was excluded at the very beginning (static flow). Whence we conclude that in the post-newtonian approximation the Tolman mass for a static, spherically symmetric, perfect fluid body, and the corresponding formula for the inertial mass, are identical.

(6): Comparison between the Dixon's mass and
the inertial (or Tolman) mass

In the exact theory of the equations of motion developed by W.G. Dixon (see e.g. [2], and improvements in [4] and [9]), the mass of an extended body is defined by

$$P^k = M_D n^k \quad (6.1)$$

where P^k (the Dixon's four-momentum) is defined with respect to a smooth, time-like curve l (e.g. the the center-of-mass world-line) contained in the support of the energy-momentum tensor, and a smooth, time-like, unit vector field on l , n^k :

$$P^k(z, n) := \int_{\Sigma} K_a^k T^{ab} d\Sigma_b \quad (6.2)$$

According to Dixon, K_a^k is a two-point tensor which maps the tangent space at z^k (a point on l) into the tangent space at the integration point x^a connected to z by a spacelike geodesic spanning a cross section Σ of the world-tube of the body. It is a convention that quantities with indices $a, b, \dots$ are defined at x , while quantities with indices $k, l, \dots$ are defined at z (e.g. $T^{ab} = T^{ab}(x)$; $K_a^k = K_a^k(x, z)$).

Inner product of (6.1) by the unit vector n^k (with the metric $g_{kl}(z)$) gives

$$P^k n_k = -M_D$$

In the following we will evaluate the Dixon's mass M_D in the case of a static, spherically symmetric gravitational

field generated by a bounded, perfect fluid source, along the lines proposed by prof. J. Ehlers in a seminar held at the Max Planck Institut für Astrophysik in January, 1978. To this purpose let us first introduce another two-point tensor, H_a^k , which, like K , arises in the theory of the Jacobian-fields: if η is a Killing field, then its value is determined by the knowledge of

$$\eta_a, \nabla[a \eta_b] \quad (6.3)$$

at a single point z , say. Dixon has shown that propagators H and K can be defined, and the Synge's world-function $\sigma(x, z)$ with derivative at z

$$\sigma^k(z, x) = - \frac{dx^k}{d\lambda}$$

(λ being the parameter along the geodesic joining z with x) can be used, in order to make the solution of the Killing equation of the type

$$\eta^a = K^a_k \eta^k + H^a_k \sigma^l \nabla^l \eta^k. \quad (6.4)$$

One usually refers to K as to the translational propagator, and to H as to the rotational one.

Furthermore H is involved in the definition of the spin bivector

$$S^{kl}(z, x) = 2 \int G^{[kH]}_{a} T^{ab} d\Sigma_b \quad (6.5)$$

where

$$d\Sigma_b := -u_b dV = -u_b \sqrt{g^{(3)}} d^3x$$

is the volume-element, $g^{(3)}$ being the determinant of the induced metric.

We can uniquely fix the world-line l via the center-of-mass condition [9]

$$p_k s^{kl} = 0. \quad (6.6)$$

In this case it can be shown that, for a static, spherically symmetric space-time,

$$s^{kl} = 0$$

and the vector n in (6.1) is tangent to l . Moreover, let $\eta = \frac{\partial}{\partial t}$ be the hypersurface orthogonal, time-like Killing vector field. The space-like hypersurface $t = \text{const.}$ will be chosen as the cross section Σ of the world-tube of the fluid, and will be called Σ_t .

Now, let us compute

$$\int_{\Sigma_t} T^{ab} \eta_b d\Sigma_a \quad (6.7)$$

Taking into account that η satisfies (6.3) and it is then of the form (6.4), we have:

$$\begin{aligned} \int_{\Sigma_t} T^{ab} \eta_b d\Sigma_a &= \eta^k \int_{\Sigma_t} T^{ab} K_{ak} d\Sigma_b = \\ &= \eta^k p_k = M_D \eta^k n_k \end{aligned} \quad (6.8)$$

But

$$n^k = e^{U_0} \eta^k \quad (U_0 := U(z))$$

and we finally write:

$$\int_{\Sigma_t} T^{ab} \eta_a d\Sigma_b = -e^{-U_0} M_D . \quad (6.9)$$

On the other hand, since

$$T^{ab} = \mu u^a u^b + p(g^{ab} + u^a u^b) ,$$

from (6.7) we also have:

$$\begin{aligned} \int_{\Sigma_t} T^{ab} \eta_a d\Sigma_b &= - \int_{\Sigma_t} [\mu u^a u^b + p(g^{ab} + u^a u^b)] \\ &\quad \cdot u_a u_b e^{-U} dV = \\ &= - \int_{\Sigma_t} \mu \sqrt{-g} d^3x \end{aligned}$$

Hence

$$M_D = e^{U_0} \int_{\Sigma_t} \mu \sqrt{-g} d^3x . \quad (6.10)$$

We can compare (2.13) with (6.10):

$$M_T = \int_{\Sigma_t} (\mu + 3p) \sqrt{-g} d^3x \quad (5.1)$$

Considering the weak field limit we have, in isotropic coordinates,

$$M_D (1 - U_0) = \int \varrho (1 + \pi + 2U) d^3x + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (6.10a)$$

$$M_T = \int \varrho (1 + \pi + \frac{5}{2} U) d^3x + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (2.13)$$

so that

$$M_D - M_T = U_0 \int \rho d^3x - \frac{1}{2} \int \rho U d^3x + [\epsilon^4 \delta, \epsilon^2 \delta^3] \quad (6.11)$$

In (6.10) we note the factor e^{U_0} , which is essentially the red-shift factor from the center. For an incompressible fluid its numerical behaviour for various values of the total mass is studied e.g. in [6], box(23.2): the potential reaches its maximum at the center, so that from (6.11) it follows that the Dixon's mass is, for an incompressible fluid, larger than the Tolman mass.

From (6.10a) we notice that

$$(1 - U_0) M_D = \int_{\Sigma_t} T^{00} d^3x + [\epsilon^4 \delta, \epsilon^2 \delta^3] \quad (6.12)$$

Where T^{00} is the (00) component of the energy-momentum tensor in the post-newtonian approximation. (6.12) can then be interpreted saying that in this approximation the Dixon's mass is the red-shifted mass defined according to the linearized theory, that is disregarding the Maxwellian-like contributions proportional to $(\nabla U)^2$ coming from the energy of the field (see (2.11)).

It is finally worthwhile to mention the interesting recent work by Börner and Rudolph [8] on the static two-body problem, who found that relativistically dense bodies (in the first and second post-newtonian approximation) attract each other according to the Newton's law, as if each of them

had precisely the Tolman mass. Generality was also gained in that they considered anisotropic stresses, which, on the other hand, are necessary in the case of a static, two-body problem.

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Chapter III

Relativistic Gravitational Interactions between Bodies
of Finite Dimensions (Part I)

Abstract: The local equations of motion

$$T^{ab}_{;b} = 0$$

in the first post-newtonian approximation to General Relativity are formulated in a new way, which allows an equation of motion for each body of an isolated, N-body system to be obtained as a first-order, ordinary differential equation. We define a self-force, and we prove that it does not affect the overall motion of each body, like in newtonian theory. A method is proposed for representing the bilinear and trilinear post-newtonian gravitational interactions between extended bodies, in terms of diagrams. If each body is spherically symmetric and in pure translational motion, a lagrangian exists from which the equation of motion of the centers of the bodies of the N-body system can be derived. This lagrangian is identical with the EIH one.

(1): Introduction

With the publication of the General Theory of Relativity, the problem arose of describing the motion of gravitating bodies, according to the Einstein's theoretical framework. The solution found by Schwarzschild already in 1916 gave support to the theory, since it accounted for the 43"/century residual in the perihelion shift of Mercury, considered as a test particle moving along a geodesic in the Schwarzschild field of the sun. In the same year a completely different approach was proposed by a student of Lorentz, J. Droste [4], who in his doctoral dissertation presented the first hydrodynamical treatment of the problem of describing the relativistic motion of N extended liquid bodies: he obtained a lagrangian very similar to the one obtained 22 years later by Einstein, Infeld and Hoffmann [7], which is still currently used in problems of relativistic celestial mechanics! Later, the problem of the motion of bodies according to General Relativity, has been considered from two different points of view: on the one hand the Einstein's attitude that the energy-momentum tensor was an undesirable term in the field equations, so that one should work with the vacuum equations, thinking of the bodies as of singularities of the field. On the other hand, Levi-Civita, Fock and his school, and later Papapetrou, Ehlers, Hogan and Mc Crea, Spyrou and others, who considered the problem of motion starting from the analysis of the hydrodynamical properties of the bodies composing the system, which are described by an energy-momentum tensor considered as an essential part of the

field equations.

This investigation adheres to the Droste-Fock point of view: the problem we intend to study is in fact the one of deriving an equation of motion of an extended, fluid body, belonging to an isolated, self-gravitating N-body system, within the framework of the post-newtonian approximation to General Relativity, which has been reformulated in chapter 1. To this purpose, the following strategy will be used: after an introduction of the various quantities - and in particular the post-newtonian potentials - we will need (sections 2 and 3), we will consider in sect. 4 the local equations of motion

$$T^{ab}_{;b} = 0 \quad (1.1)$$

in the post-newtonian approximation, with the use of the metric and Christoffel symbols obtained from an approximate solution method of a reduced Einstein equation, discussed in chapter 1. We will then show that quantities π_α , $\sigma_{\alpha\beta}$, f_α , can be defined, such that the equation

$$\partial_t \pi_\alpha + \partial_\beta \sigma_{\alpha\beta} = f_\alpha \quad (1.2)$$

is equivalent to (1.1) if terms of post-post-newtonian order are neglected. An important property of these quantities is that they do not contain time-derivatives, so that (1.2) is a local equation of motion in a precise sense (cf. Ch.1, sect.5).

In sect.4, (1.2) is integrated over the volume of a generic body of the system, to give an equation of the form

$$\frac{dP_\alpha}{dt} = F_\alpha \quad (1.3)$$

with

$$P_{\alpha} = \int_A \pi_{\alpha} d^3x \quad ; \quad F_{\alpha} = \int_A f_{\alpha} d^3x . \quad (1.4)$$

The force F is shown to consist of the sum of two classes of terms: one class describes the mutual interactions of the various fluid elements of one particular body (A , in this case), while the second class the interaction of the elements of the body A with the remaining bodies of the system.

Referring to the first class of terms as to a "self-force", and to the second one as to an "external force", it is shown that the self force is negligibly small, so that its effect on the equation of motion (1.3) is zero.

Section 5 is dedicated to the discussion of a post-newtonian potential, which is non linear in the mass-density.

In section 6 the bodies are assumed to be spherically symmetric and in pure translational motion, and a Lagrangian is constructed from which (1.3) can be derived as Euler-Lagrange equation. The Lagrangian is found to be identical to the well-known Einstein-Infeld-Hoffmann (EIH) one, which was derived for spherically symmetric field singularities.

In section 7 the results are discussed and compared with previous investigations.

(2): The Post-newtonian Potentials

As it first appeared clear from the pioneering work of Chandrasekhar [1] on post-newtonian hydrodynamics, in the post-newtonian framework one is lead to define two types of potentials, depending on whether they are linear or non linear on the density of matter ϱ . Although some of them have already been introduced in the previous chapters, we prefer to write all of them together, as follows:

Linear Potentials:

the newtonian potential is defined, in our unities, by

$$8\pi U(x, t) := \int \varrho' |x - x'|^{-1} d^3x \quad (2.1)$$

(where $\varrho' := \varrho(x', t)$), and it solves

$$\nabla^2 U = -\varrho/2.$$

Beside it we introduce the vector potential:

$$8\pi W_\alpha(x, t) := \int \varrho' v'_\alpha |x - x'|^{-1} d^3x \quad (2.2)$$

(where $v'_\alpha = v_\alpha(x', t)$ is the three velocity field of the matter), which solves

$$\nabla^2 W_\alpha = -\varrho v_\alpha/2,$$

and the tensor potential:

$$8\pi U_{\alpha\beta\gamma}(x, t) := \int \varrho' v'_\alpha \frac{(x_\beta - x'_\beta)(x_\gamma - x'_\gamma)}{|x - x'|^3} d^3x'. \quad (2.3)$$

Non Linear Potential:

following Chandrasekhar [1], it is defined by

$$8\pi \Phi(x,t) := \int \rho' \varphi' |x - x'|^{-1} d^3x' \quad (2.4)$$

where:

$$\varphi(x,t) := v^2 + U + \frac{3p}{2\rho} + \frac{\pi}{2}$$

(p is the pressure, π is the internal specific energy),
and it solves

$$\nabla^2 \Phi = -\rho \varphi/2 \quad (2.5)$$

It is non linear since also a potential (the newtonian one) acts as source term for it.

Note that, owing to the chosen unities, the potentials are dimensionless: as discussed in chapter 1, we can choose the typical value of the newtonian potential at the surface of the bodies (compactness parameter) as small, dimensionless parameter, ε^2 , measuring the weakness of the gravitational field.

Even the 3-velocity is dimensionless and we can likewise write, for its typical value:

$$v = [\delta] \text{ with } \delta \ll 1$$

As in the previous chapters, we will adopt the convention that squared brackets indicate a formal estimate of the order of magnitude:

$$U = [\varepsilon^2]; \quad W_\alpha = [\varepsilon^2 \delta]; \quad U_{\alpha, \rho, \gamma} = [\varepsilon^2 \delta]; \quad \Phi = [\varepsilon^4, \varepsilon^2 \delta^2].$$

We see that the vector, tensor and non linear potentials are quantities "smaller" than the newtonian potential; and in fact they start coming into play only in the first

post-newtonian approximation.

Note that the density ϱ entering the definitions (2.1, 2.4) does not satisfy the three-dimensional form of the law of conservation of the baryonic number. Instead the "coordinate density"

$$\varrho^* := \sqrt{-g} u^0 \varrho$$

clearly satisfies

$$\partial_t \varrho^* + \partial_\mu (\varrho^* v_\mu) = 0 \quad (2.6)$$

In post-newtonian approximation it is a standard result

$$\varrho^* = \varrho \left(1 + \frac{v^2}{2} + 3U \right) + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (2.7)$$

so that

$$\varrho^* - \varrho = [\varepsilon^4, \varepsilon^2 \delta^2]$$

Whence we could also write (2.2), (2.3) and (2.4) as

$$8\pi W_\alpha = \int \varrho^{*'} v'_\alpha |\mathbf{x} - \mathbf{x}'|^{-1} d^3 x' + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (2.2a)$$

$$8\pi U_{\alpha \cdot \beta \gamma} = \int \varrho^{*'} v'_\alpha \frac{(x_\beta - x'_\beta)(x_\gamma - x'_\gamma)}{|\mathbf{x} - \mathbf{x}'|^3} d^3 x' + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (2.3a)$$

$$8\pi \Phi = \int \varrho^{*'} \varphi' |\mathbf{x} - \mathbf{x}'|^{-1} d^3 x' + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (2.4a)$$

We will see later (sect. 3) that these expressions of the post-newtonian potentials can be equally well used as the original ones. In the post-newtonian equations of motion in fact, the dropped terms are of order

$$[\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \quad (2.8)$$

and the post-newtonian potentials come into play always

multiplied by a factor of order $[\varepsilon^2 \delta]$ (e.g. ϱv_φ) which makes the error term in (2.2a-2.4a) of the same order of magnitude, or even smaller, than (2.8).

We notice that each integrand in (2.1 -2.4) has support in the bodies forming the system, so that, for a fixed field point, it can be split into a "self" and "external" part. Focusing attention on the body A (say), and calling the other bodies B, C, D, ..., following [8] (p. 75), or [3], we introduce the decomposition

$$\begin{aligned} U(x, t) &= \mu_a + U^{(a)} \\ W_\alpha(x, t) &= w_{(a)\alpha} + W^{(a)} \\ U_{\alpha \cdot \beta \gamma}(x, t) &= \mu_{(a)\alpha \cdot \beta \gamma} + U_{\alpha \cdot \beta \gamma}^{(a)} \\ \Phi(x, t) &= \Phi^{(a)} + \Phi^{(ext)} \end{aligned} \quad (2.9)$$

($x \in \text{supp}(A)$), where the self and external parts of the potentials are respectively defined by:

$$8\pi \mu_a := \int_A \varrho' |x - x'|^{-1} d^3 x', \dots, 8\pi \Phi^{(a)} := \int_A \varrho' \varphi' |x - x'|^{-1} d^3 x' \quad (2.10a)$$

$$8\pi U^{(a)} := \sum_{B \neq A} \int_B \varrho' |x - x'|^{-1} d^3 x', \dots, 8\pi \Phi^{(ext)} := \sum_{B \neq A} \int_B \varrho' \varphi' |x - x'|^{-1} d^3 x' \quad (2.10b)$$

Likewise if the field point with coordinates x belongs to the support of B, C, D, ...

The denomination "self" and "external" might be inappropriate for the non linear potential Φ , since one of the source-terms, the linear potential U , should undergo the same splitting too: we refer to sect.5 for the discussion of this point.

(3): Post-newtonian Hydrodynamics

We consider an isolated (in the sense of [6]), self gravitating system of N perfect-fluid bodies. The energy-momentum tensor has the form

$$T^{ab} = [\rho(1 + \pi) + p] u^a u^b + p g^{ab} \quad (3.1)$$

where the local four-velocity of the fluid u^a satisfies the normalization condition

$$g_{ab} u^a u^b = -1 \quad (3.2)$$

The coordinate three-velocity is defined by

$$v_\alpha := u^\alpha / u^0$$

and the red-shift factor u^0 is related to the metric and to the three-velocity by

$$u^0 = (-g_{00} - 2g_{0\alpha} v_\alpha - g_{\alpha\beta} v_\alpha v_\beta)^{-1/2} \quad (3.3)$$

The components of the metric tensor will be considered in the approximate, post-newtonian version, in harmonic coordinates, according to the method discussed in chapter 1:

$$g_{00} = - \left\{ 1 - 2U - \frac{1}{4\pi} \nabla^2 [\rho + \mu] + 6U^2 \right\} + [\varepsilon^4 \delta, \varepsilon^2 \delta^2]$$

$$g_{0\alpha} = -4W_\alpha + [\varepsilon^4, \varepsilon^2 \delta^2] \quad (3.4)$$

$$g_{\alpha\beta} = \delta_{\alpha\beta} (1 + 2U) + [\varepsilon^2 \delta]$$

The notation

$$I_{-1} [00 + \mu\mu]$$

was introduced in chapter 1 and it is related to the Chandrasekhar's notation [1] by

$$\frac{1}{4\pi} I_{-1} [00 + \mu\mu] = \\ = 4 \Phi + 4 U^2 - \chi_{,tt} + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4]$$

where χ is the "super-potential"

$$8\pi \chi(x,t) := - \int g' |x-x'| d^3x' \quad (3.5)$$

which occurs already in newtonian theory ([2], sect.10).

The post-newtonian components of the energy-momentum tensor, considered as a functional of the matter variables and of the metric, are

$$T^{00} = g(1 + v^2 + 2U + \pi) + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \\ T^{0\alpha} = T^{00} v_\alpha + \rho v_\alpha + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \\ T^{\alpha\beta} = T^{00} v_\beta + \rho(1 - 2U) \delta_{\alpha\beta} + [\varepsilon^6 \delta, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (3.6)$$

The quantities (3.4) and (3.6) are the basic ingredients in order to write the equations

$$T^{0a}_{;a} = 0 \quad (3.7)$$

$$T^{\alpha a}_{;a} = 0 \quad (3.8)$$

$$(g^{ua})_{;a} = 0 \quad (3.9)$$

in the post-newtonian approximation.

In newtonian hydrodynamics one has the four equations

$$\partial_t \varrho + \partial_\mu (\varrho v_\mu) = 0 \quad (3.10)$$

$$\partial_t (\varrho v_\alpha) + \partial_\beta (\varrho v_\alpha v_\beta + p \delta_{\alpha\beta}) = \varrho \partial_\alpha U \quad (3.11)$$

and the five variables ϱ, p, v_α . One therefore eliminates one variable by assuming an equation of state of the form $\varrho = \varrho(p)$. In post-newtonian hydrodynamics instead we take as basic equations of motion eqs. (3.8), (3.9) for the basic variables ϱ^*, v_α . For a given equation of state $\pi = \pi(\varrho)$, eq. (3.7) is implied by them. We believe this set of basic variables and this form of the equation of state as the most convenient choice at any order of approximation. The energy-momentum tensor will then be in general of the form

$$T^{ab} = T^{ab}(\varrho^*, v, g)$$

where ϱ^* satisfies a continuity equation.

Equations (3.7) and (3.9) are primarily involved in the definition of the mass of a body, and in its time-evolution (see Ch. 2). In this chapter we will basically be concerned with (3.8), which be referred to as equation of motion in the "normal form", since it follows from the field equations as a differential property of the Einstein tensor.

(4): "Normal" versus "Quasi-newtonian" form
of the Equations of Motion

As anticipated in sect.1, in this section we will bring eq. (3.8) to the "quasi-newtonian" form

$$\partial_t \pi_\alpha + \partial_\beta \sigma_{\alpha\beta} = f_\alpha \quad (1.2)$$

To this purpose, first consider that (3.8) explicitly reads

$$\begin{aligned} \partial_t T^{\alpha 0} + \partial_\beta T^{\alpha\beta} + \left\{ \begin{matrix} \alpha \\ 00 \end{matrix} \right\} T^{00} + 2 \left\{ \begin{matrix} \alpha \\ \rho 0 \end{matrix} \right\} T^{\rho 0} + \\ + \left\{ \begin{matrix} \alpha \\ \rho\delta \end{matrix} \right\} T^{\rho\delta} + \gamma_0 T^{\alpha 0} + \gamma_\beta T^{\alpha\beta} = 0 \end{aligned} \quad (4.1)$$

with

$$\gamma_b := \left\{ \begin{matrix} a \\ b \ a \end{matrix} \right\} \quad (4.2)$$

The post-newtonian Christoffel symbols we need are (see chapter 1 for derivation)

$$\begin{aligned} \left\{ \begin{matrix} \alpha \\ 00 \end{matrix} \right\} &= -U_{,\alpha} + 4 U^2_{,\alpha} - 4 W_{\alpha,t} - \frac{1}{8\pi} I_{-1,\alpha} [{}^{00} + \mu\mu] + [\mathcal{E}^4 \delta, \mathcal{E}^2 \delta^3] \\ \left\{ \begin{matrix} \alpha \\ \rho 0 \end{matrix} \right\} &= 2 (W_{\rho,\alpha} - W_{\alpha,\rho}) + \delta_{\alpha\rho} U_{,t} + [\mathcal{E}^4, \mathcal{E}^2 \delta^2] \\ \left\{ \begin{matrix} \alpha \\ \rho\delta \end{matrix} \right\} &= (\delta_{\alpha\rho} U_{,\delta} + \delta_{\alpha\delta} U_{,\rho} - \delta_{\rho\delta} U_{,\alpha}) + [\mathcal{E}^2 \delta] \end{aligned} \quad (4.3)$$

Inserting (4.2), (4.3) and the post-newtonian energy-momentum tensor (3.6) into (4.1), we obtain

$$\begin{aligned}
& \partial_t \left[\varrho (1 + v^2 + 2U + P/\varrho + \pi) v_\alpha \right] + \partial_\beta \left[\varrho (1 + v^2 + 2U + P/\varrho + \pi) v_\alpha v_\beta + \right. \\
& \quad \left. + P(1 - 2U) \delta_{\alpha\beta} \right] + \varrho \left(-U_{,\alpha} + 4U^2_{,\alpha} - 4W_{\alpha,t} - \frac{I_{-1,\alpha}}{8\pi} [00 + \mu\mu] \right) - \\
& \quad - \varrho (v^2 + 2U + \pi) U_{,\alpha} + 4\varrho v_\beta (W_{\beta,\alpha} - W_{\alpha,\beta}) + 4\varrho v_\alpha U_{,t} + \\
& \quad + 2U_{,\beta} (\varrho v_\alpha v_\beta + P \delta_{\alpha\beta}) + (\delta_{\alpha\beta} U_{,\gamma} + \delta_{\alpha\gamma} U_{,\beta} - \\
& \quad - \delta_{\gamma\beta} U_{,\alpha}) (\varrho v_\beta v_\gamma + P \delta_{\beta\gamma}) = [\varepsilon^4 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \quad (4.4)
\end{aligned}$$

Making use of the equations of motion at the newtonian order (3.10) and (3.11), of the definition (2.7) of ϱ^* , and of the Leibnitz rule for the derivatives, and replacing the right hand side of (4.4) by zero, it is matter of some algebra to show that (4.4) can be reordered into the "quasi-newtonian" form (1.2), where:

$$\begin{aligned}
\pi_\alpha &:= \varrho^* (1 + \frac{v^2}{2} - U + \pi + P/\varrho^*) v_\alpha + 4\varrho^* (v_\alpha U - W_\alpha) + \frac{1}{2} \varrho^* (W_\alpha - U_{\beta,\alpha\beta}) \\
\sigma_{\alpha\beta} &:= \pi_\alpha v_\beta + (1 + 2U) P \delta_{\alpha\beta} \\
f_\alpha &:= \varrho \partial_\alpha U - 4\varrho^* v_\mu \partial_\alpha W_\mu + 2\varrho^* (\varphi \partial_\alpha U + \partial_\alpha \Phi) + \\
&\quad + \frac{1}{2} \varrho^* v_\mu \partial_\mu (W_\alpha - U_{\beta,\alpha\beta}) \quad (4.5)
\end{aligned}$$

We remark that the expressions (4.5) do not contain time-derivatives, so that (1.2) is a local equation of motion in the same sense as the newtonian Euler equation (3.11) (cf. also discussion in sect.5 of Ch.1). From (1.4) it follows

$$\begin{aligned}
P_\alpha &= \int_A \varrho^* \left[\left(1 + \frac{v^2}{2} - U + \pi + P/\varrho^* \right) v_\alpha + 4(v_\alpha U - W_\alpha) + \right. \\
&\quad \left. + \frac{1}{2} (W_\alpha - U_{\beta,\alpha\beta}) \right] d^3x \quad (4.6)
\end{aligned}$$

$$F_\alpha := \int_A \varrho \partial_\alpha U d^3x + \int_A \varrho^* \left[-4 v_\mu \partial_\alpha W_\mu + 2 (\varphi \partial_\alpha U + \Phi_{,\alpha}) + \frac{1}{2} v_\mu \partial_\mu (W_\alpha - U_{\beta,\alpha\beta}) \right] d^3x \quad (4.7)$$

These expressions can be significantly simplified, taking into account the decompositions (2.9) of the potentials into a self and internal part.

We first prove the following lemmas :

Lemma 1: in newtonian theory the self force vanishes identically :

$$\int_A \varrho \partial_\alpha u_a d^3x = 0$$

Proof: by definition we have

$$\int_A \varrho \partial_\alpha u_a d^3x = \iint_A \varrho \varrho' \partial_\alpha |x-x'|^{-1} d^3x d^3x' \equiv 0$$

since the integrand must be symmetric in the exchange of x and x' .

Lemma 2:

$$\int_A \varrho (v_\alpha u_a - w_{(a)\alpha}) d^3x = 0$$

The proof is obvious on the basis of the definition of $w_{(a)\alpha}$.

Lemma 3:

$$\int_A \varrho v_\mu \partial_\alpha w_{(a)\mu} d^3x = 0$$

Proof: by definition we have

$$\int_A \varrho v_\mu \partial_\alpha w_{(a)\mu} d^3x = \iint_A \varrho v_\mu \varrho' v'_\mu \partial_\alpha |x-x'|^{-1} d^3x d^3x' \equiv 0$$

since the integrand is symmetric in x and x' .

In the same way one verifies

Lemma 4:

$$\int_A g (\varphi \partial_\alpha u_\alpha + \partial_\alpha \Phi^{(3)}) d^3x = 0$$

Lemma 5:

$$\int_A g v_\mu \partial_\mu \left(w_{(\alpha)}^\alpha - \frac{\mu}{(\alpha)} \beta \cdot \alpha \beta \right) d^3x = [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5]$$

Proof: consider again the super-potential (3.5)

$$8\pi \chi(x, t) = - \int g' |x - x'| d^3x' \quad (3.5)$$

and assuming x to be in the support of the body A , let us split (3.5) into a self and external part

$$\chi(x, t) = \chi^{(s)} + \chi^{(ext)}$$

with

$$8\pi \chi^{(s)} := - \int_A g' |x - x'| d^3x' ; \quad 8\pi \chi^{(ext)} := - \sum_{B \neq A} \int_B g' |x - x'| d^3x'.$$

Using the hydrodynamical equations at the newtonian order (3.10) and (3.11), it is straightforward to show that

$$\chi_{, \alpha t} = W_\alpha - U_{\beta \cdot \alpha \beta} + [\varepsilon^4, \varepsilon^2 \delta^2]$$

Also

$$\begin{aligned} \int_A g v_\mu \partial_\mu \left(w_{(\alpha)}^\alpha - \frac{\mu}{(\alpha)} \beta \cdot \alpha \beta \right) d^3x &= \int_A g v_\mu \chi_{, \alpha \mu t} d^3x + \\ &+ [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] = \\ &= \int_A g v_\mu \partial_\alpha \left(w_{(\alpha)}^\mu - \frac{\mu}{(\alpha)} \beta \cdot \mu \beta \right) d^3x + [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \end{aligned}$$

The first integral vanishes because of Lemma 3, and the same

happens for the second integral, as is easy to see.

Introducing now the decompositions of the potentials (2.9) into the expression for the momentum (4.6), we see that, by virtue of Lemmas 1 and 2, P_α reduces to

$$P_\alpha = P_\alpha^{(s)} + P_\alpha^{(ext)} \quad (4.8)$$

with

$$P_\alpha^{(s)} := \int_A \varrho^* \left[\left(1 + \frac{v^2}{2} - u_a + \pi + \frac{p}{\varrho^*} \right) v_\alpha + \frac{1}{2} \left(w_{(a)}^\alpha - \frac{u}{(a)} \beta \cdot \alpha \beta \right) \right] d^3x \quad (4.9)$$

$$P_\alpha^{(ext)} := \int_A \varrho^* \left[3 v_\alpha U^{(a)} - 4 W_\alpha^{(a)} + \frac{1}{2} \left(W_\alpha^{(a)} - U_{\beta \cdot \alpha \beta}^{(a)} \right) \right] d^3x \quad (4.10)$$

Likewise for the expression (4.7) of the post-newtonian force F_α , which can be decomposed into a self-force

$$F_\alpha^{(s)} := \int_A \left\{ \varrho \partial_\alpha u_a + \varrho^* \left[-4 v_\mu w_{(a)}^\mu{}_{,\alpha} + 2 \left(\varphi u_{a,\alpha} + \Phi_{,\alpha}^{(a)} \right) + \frac{1}{2} v_\mu \left(w_{(a)}^\mu{}_{,\alpha} - \frac{u}{(a)} \beta \cdot \alpha \beta \right)_{,\mu} \right] \right\} d^3x$$

and an external force

$$F_\alpha^{(ext)} := \int_A \left\{ \varrho \partial_\alpha U^{(a)} + \varrho^* \left[-4 v_\mu W_{\alpha,\mu}^{(a)} + 2 \left(\varphi U_{,\alpha}^{(a)} + \Phi_{,\alpha}^{(ext)} \right) + \frac{1}{2} v_\mu \left(W_\alpha^{(a)} - U_{\beta \cdot \alpha \beta}^{(a)} \right)_{,\mu} \right] \right\} d^3x \quad (4.11)$$

so that

$$F_\alpha = F_\alpha^{(s)} + F_\alpha^{(ext)}$$

By virtue of Lemmas 1,3,4,5, we conclude that $F_\alpha^{(s)}$ is negligibly small in the post-newtonian approximation, so

that its contribution in the equation of motion (1.2) can be substituted by zero.

Eq. (1.2) can then be rewritten in the form

$$\frac{d}{dt} (p_{\alpha}^{(s)} + p_{\alpha}^{(ext)}) = F_{\alpha}^{(ext)} . \quad (4.12)$$

(5): Some Remarks on the Non-Linear Force

From (4.11) we see that the post-newtonian force acting on the body A is bilinear on the sources (just as in newtonian theory) except for the third term, which will be referred to as the non-linear force:

$$F_{\alpha}^{(n.l.)} := 2 \int_A g \left[\varphi \partial_{\alpha} U^{(a)} + \partial_{\alpha} \Phi^{(ext)} \right] d^3 x \quad (5.1)$$

This section is dedicated to the analysis of the structure of the non-linear force.

To this purpose, let us first consider the splitting

$$\varphi = \varphi_0 + U$$

so that

$$\varphi_0 := v^2 + \frac{\Pi}{2} + \frac{3\rho}{2g}$$

Hence

$$\begin{aligned} & \int g \left(\varphi \partial_{\alpha} U^{(a)} + \partial_{\alpha} \Phi^{(ext)} \right) d^3 x \\ &= \int_A d^3 x g \left(\varphi_0 + u_a + U^{(a)} \right) \sum_{B \neq A} \int_B d^3 x' g' \partial_{\alpha} |x - x'|^{-1} + \\ &+ \int_A d^3 x g \sum_{B \neq A} \int_B g' \left(\varphi'_0 + u'_b + U^{(b)} \right)' \partial_{\alpha} |x - x'|^{-1} d^3 x' \end{aligned} \quad (5.2)$$

Inserting the definitions for the self and external potentials, we reorder (5.2) in the form

$$\begin{aligned}
& \int_A d^3x \varphi \left(\varphi_0 \sum_{B \neq A} \int_B d^3x' \varphi' + \sum_{B \neq A} \int_B d^3x' \varphi' \varphi_0' \right) \partial_\alpha |x-x'|^{-1} + \\
& + \int_A d^3x \varphi \left(u_a \sum_{B \neq A} \int_B d^3x' \varphi' + \sum_{B \neq A} \int_B d^3x' \varphi' u_b' \right) \partial_\alpha |x-x'|^{-1} + \\
& + \left(\frac{1}{8\pi} \right)^2 \int_A d^3x \varphi \sum_{B \neq A} \int_B d^3x' \varphi' \left(\int_B \varphi'' |x-x'|^{-1} d^3x'' + \sum_{\substack{C \neq A \\ C \neq B}} \int_C \varphi'' |x-x''|^{-1} d^3x'' \right) \partial_\alpha |x-x'|^{-1} + \\
& + \left(\frac{1}{8\pi} \right)^2 \int_A d^3x \varphi \sum_{B \neq A} \int_B d^3x' \varphi' \left(\int_A \varphi'' |x'-x''|^{-1} d^3x'' + \sum_{\substack{C \neq A \\ C \neq B}} \int_C \varphi'' |x'-x''|^{-1} d^3x'' \right) \partial_\alpha |x-x'|^{-1}.
\end{aligned} \tag{5.3}$$

We can rewrite only the last two terms in the sum (5.3)

$$\begin{aligned}
(5.3) = \dots & + \left(\frac{1}{8\pi} \right)^2 \int_A d^3x \varphi \sum_{B \neq A} \int_B d^3x' \varphi' \left(\int_A d^3x'' \varphi'' |x'-x''|^{-1} + \int_B d^3x'' \varphi'' |x-x''|^{-1} \right) \cdot \\
& \cdot \partial_\alpha |x-x'|^{-1} + \\
& + \left(\frac{1}{8\pi} \right)^2 \int_A d^3x \varphi \sum_{B \neq A} \int_B d^3x' \varphi' \sum_{\substack{C \neq A \\ C \neq B}} \left(\int_C d^3x'' \varphi'' |x'-x''|^{-1} + \right. \\
& \left. + \int_C d^3x'' \varphi'' |x-x''|^{-1} \right) \partial_\alpha |x-x'|^{-1} \tag{5.4}
\end{aligned}$$

We will come back later to these terms, under the assumption that the bodies interact only through the monopole part of their potentials: in any case it is already clear that the last term in (5.4) is peculiar of three body interaction.

We consider now the terms in (5.3), or (5.4) significant for the two body interaction: assuming for simplicity a two body system (A,B), we can recognize the interaction between the two extended bodies with the help of the following diagrams,

which show a very typical way of gravitational coupling of the various forms of energy density, in the post-newtonian approximation:

(i)

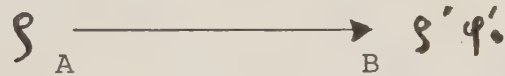
$$\int_A d^3x \, g \varphi_0 \int_B d^3x' \, g' \partial_\alpha |x - x'|^{-1}$$



The density g' in B couples to $g\varphi_0$ in A via the propagator $\partial_\alpha |x - x'|^{-1}$ (oriented and continuous line).

(ii)

$$\int_A d^3x \, g \int_B d^3x' \, g' \varphi'_0 \partial_\alpha |x - x'|^{-1}$$



The same as in (i), but with the roles of the two densities reversed.

(iii)

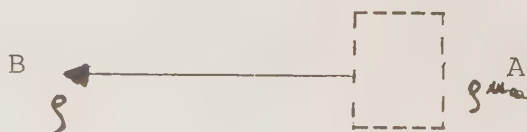
$$\int_A d^3x \, g \int_B d^3x' \, g' u'_B \partial_\alpha |x - x'|^{-1}$$



The density in A couples to the self energy in B, represented by a loop (dashed unoriented line).

(iv)

$$\int_A d^3x \, g u_A \int_B d^3x' \, g' \partial_\alpha |x - x'|^{-1}$$



The same as in (iii), but with the roles of the two densities reversed.

$$(v) \quad \int_A d^3x \, \varrho \, U^{(a)} \int_B d^3x' \, \varrho' \, \partial_\alpha |x-x'|^{-1}$$



The density ϱ' in B couples with the energy-density $\varrho U^{(a)}$ coming from the interaction of A with B.

(vi)



The same as in (v), but with the roles of the densities ϱ , $\varrho U^{(a)}$ reversed with the ones of ϱ , $\varrho' U'^{(b)}$.

Furthermore, for a system composed with three or more bodies A, B, C, ..., there will be the additional diagrams showing the coupling of the density ϱ of A with the energy-density coming from the interaction between B and C, C and D, ... and so on.

(6): Study of the N-body Problem

This section is dedicated to the study of eq. (4.12) in the case of two bodies (the extension to $N > 2$ is straightforward and will be given at the end of this section) under the following assumptions:

- a) in each body there is one point with respect to which the body is spherically symmetric.
- b) each body is in pure translational motion.
- c) the radius of each body is small compared with the separation from the other.

Assumption b) means that there exists a local, non rotating comoving frame attached at the symmetry point introduced by assumption a), and unique up to rotations, with respect to which each point of the given body keeps its orientation fixed in time. Assumption c) implies that in a multipole expansion of the external potentials, the first (monopole) term dominates all the others.

Furthermore, if each body is in stable equilibrium, then the self-potential and the pressure satisfy the virial theorem in the local system $(\tilde{\xi}, \tilde{t})$ attached to the local comoving frame

$$\frac{1}{2} \int_A \rho \tilde{\mu}_a d^3 \tilde{\xi} = 3 \int_A \rho d^3 x + [\epsilon^4 \delta, \epsilon^2 \delta^3] \quad (6.1)$$

This relation remains true in the global, inertial frame, up to terms of order $[\varepsilon^4 \delta, \varepsilon^2 \delta^3]$.

Assumption a) also implies the existence of no preferred direction in the comoving frame:

$$\langle \frac{(\tilde{\xi}_\alpha - \tilde{\xi}'_\alpha)(\tilde{\xi}_\beta - \tilde{\xi}'_\beta)}{|\tilde{\xi} - \tilde{\xi}'|^2} \rangle = \frac{1}{3} \delta_{\alpha\beta} \quad (6.2)$$

Whence the quantity $\tilde{u}_{(\alpha)\alpha\beta}$ satisfies

$$\tilde{u}_{(\alpha)\alpha\beta} := \int_A \tilde{g}' \frac{(\tilde{\xi}_\alpha - \tilde{\xi}'_\alpha)(\tilde{\xi}_\beta - \tilde{\xi}'_\beta)}{|\tilde{\xi} - \tilde{\xi}'|^3} d^3 \tilde{\xi}' = \frac{1}{3} \delta_{\alpha\beta} \tilde{u}_a \quad (6.3)$$

and in the global inertial frame

$$\tilde{u}_{(\alpha)\alpha\beta} = \frac{1}{3} \delta_{\alpha\beta} u_a + [\varepsilon^2 \delta] \quad (6.4)$$

The assumption that the motion is purely translational implies

$$\dot{\tilde{\xi}} = 0 \quad (6.5)$$

that is

$$\underline{y}(x, t) = \begin{cases} \underline{\dot{a}}(t) & \text{if } x \in \text{supp}(A) \\ \underline{\dot{b}}(t) & \text{if } x \in \text{supp}(B) \end{cases} \quad (6.6)$$

$\underline{a}$ and $\underline{b}$ are the locations of the centers of the bodies A and B respectively, and $\underline{\dot{a}}$, $\underline{\dot{b}}$ their 3-velocities.

From (6.4) and (6.6) it follows

$$\begin{aligned} w_{(\alpha)\alpha} &= \dot{a}_\alpha u_a + [\varepsilon^4, \varepsilon^2 \delta^2] \\ u_{(\alpha)\beta \cdot \alpha\beta} &= \frac{1}{3} \dot{a}_\alpha u_a + [\varepsilon^4, \varepsilon^2 \delta^2] \end{aligned} \quad (6.7)$$

Finally, the assumption of small (radius)/(distance) ratio allows the external potentials to be written as follows:

$$\begin{aligned} 8\pi U^{(a)} &= \frac{m_B}{R} + O(R^{-2}) \quad ; \quad 8\pi W_{\alpha}^{(a)} = \frac{m_B}{R} \dot{b}_{\alpha} + O(R^{-2}) \\ 8\pi U_{\beta \cdot \alpha \beta}^{(a)} &= m_B (\underline{\dot{b}} \cdot \underline{R}) \frac{R_{\alpha}}{R^3} + O(R^{-2}) \end{aligned} \quad (6.8)$$

where $\underline{R} = \underline{a} - \underline{b}$.

Consider the self part of the momentum (4.9), concerning the body A. Introducing the inertial mass

$$m_A := \int_A g^* \left(1 - \frac{1}{2} u_a + \pi \right) d^3x \quad (6.9)$$

which in chapter 2 was proved to be constant for non-dissipative processes, and using (6.1), (6.6), (6.7) and (6.9), we obtain:

$$P_{\alpha}^{(s)} = m_A \left(1 + \frac{1}{2} \dot{a}^2 \right) \dot{a}_{\alpha} \quad (6.10)$$

or equivalently

$$\begin{aligned} P_{\alpha}^{(s)} &= \\ &= \frac{\partial}{\partial \dot{a}_{\alpha}} \left[m_A \left(1 + \frac{\dot{a}^2}{4} \right) \frac{\dot{a}^2}{2} + m_B \left(1 + \frac{\dot{b}^2}{4} \right) \frac{\dot{b}^2}{2} \right] \end{aligned} \quad (6.11)$$

Likewise for the external part (4.10) of the momentum, we have, making use of (6.8),

$$\begin{aligned} 8\pi P_{\alpha}^{(ext)} &= (3 \dot{a}_{\alpha} - 4 \dot{b}_{\alpha}) \frac{m_A m_B}{R} + \\ &+ \frac{1}{2} m_A m_B \dot{b}_{\beta} \frac{\partial^2 R}{\partial a_{\alpha} \partial a_{\beta}} + O(R^{-2}) \end{aligned} \quad (6.12)$$

or also

$$8\pi P_{\alpha}^{(ext)} = \frac{\partial}{\partial \dot{a}_{\alpha}} \left[\frac{m_A m_B}{2R} (3 \dot{a}^2 + 3 \dot{b}^2 - 8 \dot{a} \cdot \dot{b}) + \right. \\ \left. + \frac{1}{2} m_A m_B \dot{a}_{\alpha} \dot{b}_{\beta} \frac{\partial^2 R}{\partial a_{\alpha} \partial a_{\beta}} \right] + O(R^{-2}) \quad (6.13)$$

As an intermediate result we then write

$$P_{\alpha} = \frac{\partial}{\partial \dot{a}_{\alpha}} \left[m_A \left(1 + \frac{\dot{a}^2}{4} \right) \frac{\dot{a}^2}{2} + m_B \left(1 + \frac{\dot{b}^2}{4} \right) \frac{\dot{b}^2}{2} + \right. \\ \left. + \frac{m_A m_B}{16\pi R} (3 \dot{a}^2 + 3 \dot{b}^2 - 8 \dot{a} \cdot \dot{b} + \frac{1}{2} \dot{a}_{\mu} \dot{b}_{\nu} \frac{\partial^2 R}{\partial a_{\mu} \partial a_{\nu}}) \right] + \\ + O(R^{-2}) \quad (6.14)$$

Now we need the force: to this purpose (4.11) will be evaluated term by term. The newtonian force is first considered. It can be expressed in terms of φ^* instead of φ , inverting (2.7): dropping the error terms we have

$$\varphi = \varphi^* \left(1 - \frac{v^2}{2} - 3U \right) \quad (6.15)$$

It then follows

$$8\pi \int_A \varphi \partial_{\alpha} U^{(a)} d^3x = \int_A d^3x \int_B d^3x' \varphi^* \left(1 - \frac{1}{2} \dot{a}^2 - 3U \right) \cdot \\ \cdot \varphi^{*'} \left(1 - \frac{1}{2} \dot{b}^2 - 3U' \right) \partial_{\alpha} |x - x'|^{-1} = \\ = - \left[m_A^* m_B^* \left(1 - \frac{\dot{a}^2}{2} - \frac{\dot{b}^2}{2} \right) - 6 \left(\varepsilon_A m_B + \varepsilon_B m_A \right) \frac{R_{\alpha}}{R^2} + \right. \\ \left. + 3 m_A m_B (m_A + m_B) \frac{R_{\alpha}}{8\pi R^3} \right] + O(R^{-3}) \quad (6.16)$$

Here , the notation

$$m_A^* := \int_A \varrho^*(1+\pi) d^3x \quad ; \quad \mathcal{E}_A = \frac{1}{2} \int_A \varrho^* u_a d^3x$$

has been temporally introduced, so that (6.9) becomes

$$m_A = m_A^* - \mathcal{E}_A \quad (6.17)$$

Likewise for B.

Eq. (6.16) can be rewritten in the form

$$\begin{aligned} 8\pi \int_A \varrho \partial_\alpha U^{(a)} d^3x &= \frac{\partial}{\partial a_\alpha} \left\{ \frac{m_A^* m_B^*}{R} \left(1 - \frac{\dot{a}^2}{2} - \frac{\dot{b}^2}{2} \right) - \right. \\ &\quad \left. - \frac{6}{R} (\mathcal{E}_A m_B + \mathcal{E}_B m_A) - \frac{3}{2} \frac{m_A m_B}{8\pi R^2} (m_A + m_B) \right\} + O(R^{-4}) \end{aligned} \quad (6.18)$$

Next consider that

$$\begin{aligned} -4 \int_A \varrho^* v_\mu \partial_\alpha W_\mu^{(a)} d^3x &+ \frac{1}{2} \int_A \varrho^* v_\mu \partial_\mu (W_\alpha^{(a)} - U_{\beta\alpha\beta}^{(a)}) d^3x = \\ &= - \frac{\partial}{\partial a_\alpha} \left[\frac{4 m_A m_B}{8\pi R} (\dot{a} \cdot \dot{b}) - \frac{m_A m_B}{16\pi R} \dot{a}_\mu \dot{b}_\nu \left(S_{\mu\nu} - \frac{R_\mu R_\nu}{R^2} \right) \right] + \\ &\quad + O(R^{-4}) \end{aligned} \quad (6.19)$$

Once again with the use of the virial theorem, the last two terms of the force can be reduced:

$$\begin{aligned} 2 \int_A \varrho^* \varphi \partial_\alpha U^{(a)} d^3x &= \frac{1}{8\pi} \frac{\partial}{\partial a_\alpha} \left[2 \frac{m_A m_B}{R} \dot{a}^2 + 5 \frac{\mathcal{E}_A m_B}{R} + \right. \\ &\quad \left. + \frac{m_B^2 m_A}{8\pi R^2} \right] + O(R^{-4}) \\ 2 \int_A \varrho^* \partial_\alpha \Phi^{(int)} d^3x &= \frac{1}{8\pi} \frac{\partial}{\partial a_\alpha} \left[2 \frac{m_A m_B}{R} \dot{b}^2 + 5 \mathcal{E}_B m_A + \right. \\ &\quad \left. + \frac{m_A^2 m_B}{8\pi R^2} \right] + O(R^{-4}) \end{aligned} \quad (6.20)$$

Collecting (6.18), (6.19) and (6.20), the second intermediate result is

$$8\pi F_\alpha = m_A m_B \frac{\partial}{\partial a_\alpha} \left\{ \frac{1}{2R} \left[3\dot{a}^2 + 3\dot{b}^2 - 8\dot{\underline{a}} \cdot \dot{\underline{b}} + \right. \right. \\ \left. \left. + \dot{a}_\mu \dot{b}_\nu \left(\delta_{\mu\nu} - \frac{R_\mu R_\nu}{R^2} \right) \right] + \frac{1}{R} - \frac{m_A + m_B}{16\pi R^2} \right\} + O(R^{-4}) \quad (6.21)$$

The expressions (6.14) and (6.21) lead to the Lagrangian

$$L(a, \dot{a}, b, \dot{b}) = m_A \left(1 + \frac{\dot{a}^2}{4} \right) \frac{\dot{a}^2}{2} + m_B \left(1 + \frac{\dot{b}^2}{4} \right) \frac{\dot{b}^2}{2} + \\ + \frac{m_A m_B}{16\pi R} \left(3\dot{a}^2 + 3\dot{b}^2 - 7\dot{\underline{a}} \cdot \dot{\underline{b}} - \frac{(\dot{\underline{a}} \cdot \underline{R})(\dot{\underline{b}} \cdot \underline{R})}{R^2} \right) + \\ + \frac{m_A m_B}{8\pi R} - \frac{1}{2} \frac{m_A m_B}{(8\pi)^2} \frac{m_A + m_B}{R^2} + O(R^{-3}) \quad (6.22)$$

which is the Lagrangian introduced in 1938 by Einstein, Infeld and Hoffmann, provided that the mass introduced in (6.9) for a perfect fluid, extended body, can be identified with the mass defined in the EIH framework for field-singularities.

Generalization to the N-body Lagrangian is straightforward: one has simply to consider the last term in (5.4), significant for N-body systems with $N \geq 3$; under the assumption c) it takes the form

$$(8\pi)^2 F_\alpha^{(3)} = \frac{1}{2} m_A \frac{\partial}{\partial a_\alpha} \sum_{\substack{A \neq B \neq C \\ A \neq C}} m_B m_C \left(\frac{1}{|b-c||a-b|} + \right. \\ \left. + \frac{1}{|a-b||a-c|} + \frac{1}{|c-b||a-c|} \right) + O(R^{-4})$$

so that the additional term for (6.22) is

$$\frac{1}{6} \sum_{\substack{A \neq B \neq C \\ A \neq C}} \frac{m_A m_B m_C}{(8\pi)^2} \left(\frac{1}{|a-b||a-c|} + \frac{1}{|a-c||c-b|} + \frac{1}{|c-b||a-b|} \right) + O(R^{-3})$$

Symmetrising the double and triple sums, the final result is

$$\begin{aligned} L = & \sum_A m_A \left(1 + \frac{\dot{a}^2}{4} \right) \frac{\dot{a}^2}{2} + \frac{1}{8\pi} \cdot \frac{1}{4} \sum_{A \neq B} \frac{m_A m_B}{|a-b|} \cdot \\ & \cdot \left(3\dot{a}^2 + 3\dot{b}^2 - 7\dot{a} \cdot \dot{b} - \frac{\dot{a} \cdot (\underline{a}-\underline{b}) \dot{b} \cdot (\underline{a}-\underline{b})}{|a-b|^2} \right) + \\ & + \frac{1}{8\pi} \cdot \frac{1}{2} \sum_{A \neq B} \frac{m_A m_B}{|a-b|} - \frac{1}{(8\pi)^2} \cdot \frac{1}{4} \sum_{A \neq B} \frac{m_A m_B}{|a-b|^2} (m_A + m_B) + \\ & + \frac{1}{(8\pi)^2} \cdot \frac{1}{6} \sum_{\substack{A \neq B \neq C \\ A \neq C}} m_A m_B m_C \left(\frac{1}{|a-b||a-c|} + \right. \quad (6.23) \\ & \left. + \frac{1}{|a-b||c-b|} + \frac{1}{|a-c||b-c|} \right) + O(R^{-3}) \end{aligned}$$

(7): Discussion

In this chapter the equations of post-newtonian hydrodynamics have been used in order to derive an equation of motion for an extended body moving in the gravitational field generated by itself and by other similar bodies forming an isolated system. Writing the local equations of motion (1.3) and proving that the self part in the post-newtonian force is zero, was of basic importance for the development of the approach. We believe (4.12) to be most useful for dealing with the motion of a body in a N-body system, whereas the other available formulation of the equations of motion, due to Landau and Lifshitz

$$\left[(-g) (T^{ab} + t^{ab}) \right]_{,b} = 0$$

is most useful for obtaining ten "first integrals" (conservation of the three components of the linear and angular momentum, conservation of the total energy and uniform motion of the center of inertia of the system), and therefore applies only for a "one-body system", that is a body non interacting with other bodies.

In the present treatment of the post-newtonian equations of motion for extended bodies, we have used some ideas of Fock [8]. However, our definitions and results differ in several important ways from those of Fock. Apart from the method of approximation which was used in chapter 1 in

order to derive the post-newtonian metric, a first important difference with respect to Fock's results arises in the definition of the momentum of an extended body: comparison⁽⁺⁾ of (4.9) with (76.03) in [8] reveals that the two quantities are different in a non trivial way. We use $\mathfrak{g}^*$ as density, while Fock uses $\mathfrak{g}$. This means that our definitions differ from Fock's by a non negligible quantity of post-newtonian order.

Further Fock defines the mass as in the newtonian approximation

$$\int_A \mathfrak{g} d^3x \quad (7.1)$$

and treats it as a constant, while it is not: the equation of motion $T^{0a}_{;a} = 0$ implies that the time-derivative of (7.1) cannot be neglected in comparison with other post-newtonian quantities. The same equation instead implies (see chapter 2) that the mass defined in (6.9) is constant in time up to terms of post-post-newtonian order, if there are no dissipative processes. We also notice that our (6.9) is different from Fock's (76.18). The reason of all such differences probably lies in the form of Fock's post-newtonian metric (see (68.43)). Surprisingly enough however, the result Fock obtains for the N-spheres system is identical to our (6.23) and to the Einstein-Infeld-Hoffmann's one, who used field-singularities as

(+) It might be useful to recall that

$$w^{(\text{Fock})} = -\frac{1}{2} \chi^{(\text{ours})}$$

model for the bodies. The present investigation then shows again that the use of field-singularities is equivalent to the use of spherical fluid balls.

Generalization to the case of rotating bodies should also be mentioned, by Haywood [9]. However no one of these authors stressed the need of giving an at least formal account of the errors, as it should be done in any approximation scheme. Hogan and Mac Crea [10] also treated the problem of the motion of extended bodies, putting particular emphasis in the specification of the neglected quantities, and obtaining for the N-spheres system a result which agrees with our (6.23). There is some similarity in the purposes of their work and of this investigation. In fact they used the Synge's method of approximate solution of Einstein equation, in the same way as we took advantage, in chapter 1, of the more recent Ehler's approach. However the technique which has been developed here for the equations of motion of N extended bodies is definitely different from the one in [10], so that (6.23) must be considered an independent confirmation of the results in [10].

The problem of the errors in approximate equations of motion was also emphasized by Ehlers and collaborators [5], and by Contopoulos and Spyrou [3], who gave detailed account of the neglected terms. Later Spyrou, in his remarkable series of papers on the N-body problem in the post-newtonian approximation, (see e.g. [11], [12]) always emphasized the need of keeping track of the orders of the omitted terms. We conclude with a few words about the usefulness of the method developed in this investigation for relativistic

celestial mechanics. The assumptions a) and b) of sect.6 about spherical symmetry and absence of intrinsic rotation have been introduced in order to make the approach as simple as possible. The scheme of investigation developed in this chapter allows these assumptions to be weakened, making the approach more general and the model more realistic for applications to astronomical systems. This will be shown in the next chapter, where the bodies will be allowed to rotate, and to have a spheroidal shape.

Appendix: derivation of (4.5)

Looking at the time-derivatives in (4.4), one has

$$\begin{aligned}
 & \partial_t \left[\varrho \left(1 + v^2 + 2U + \pi + \frac{p}{\varrho} \right) v_\alpha - 4\varrho W_{\alpha,t} + 4\varrho v_\alpha U_{,t} \right] = \\
 & = \partial_t \left[\varrho \left(1 + v^2 + 2U + \pi + \frac{p}{\varrho} \right) v_\alpha + 4\varrho (v_\alpha U - W_\alpha) \right] - 4W_\alpha (\varrho v_\mu)_{,\mu} + \\
 & + 4U \partial_\mu (\varrho v_\alpha v_\mu + p \delta_{\alpha\mu}) - 4\varrho U \partial_\alpha U + \\
 & + [\varepsilon^1 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \quad (A.1)
 \end{aligned}$$

where use has been made of the lower order equations of motion (3.10) and (3.11). Using the Leibnitz's rule in the second and third term in (A.1), one obtains

$$\begin{aligned}
 (A.1) = & \\
 & = \partial_t \left[\varrho \left(1 + v^2 + 2U + \pi + \frac{p}{\varrho} \right) v_\alpha + 4\varrho (v_\alpha U - W_\alpha) \right] - \\
 & - 4\varrho v_\mu \partial_\mu W_\alpha - 4U_{,\mu} (\varrho v_\alpha v_\mu + p \delta_{\alpha\mu}) - 4\varrho U \partial_\alpha U - \\
 & - \left[4\varrho v_\mu W_\alpha - 4U (\varrho v_\alpha v_\mu + p \delta_{\alpha\mu})_{,\mu} \right] \quad (A.2)
 \end{aligned}$$

Reconsidering (4.4)

$$\begin{aligned}
 & \partial_t \left[\varrho \left(1 + v^2 + 2U + \pi + \frac{p}{\varrho} \right) v_\alpha \right] + \partial_\beta \left[\varrho \left(1 + v^2 + 2U + \pi + \frac{p}{\varrho} \right) \right. \\
 & \cdot v_\alpha v_\beta + p(1 - 2U) \delta_{\alpha\beta} \left. \right] - \varrho U_{,\alpha} + 4\varrho U_{,\alpha}^2 - 4\varrho W_{\alpha,t} - \\
 & - \frac{\varrho}{8\pi} I_{-1,\alpha} [00 + \mu\mu] - 2\varrho \left(v^2 + U + \frac{\pi}{2} + \frac{3p}{2\varrho} \right) U_{,\alpha} + \\
 & + 8\varrho v_\beta W_{[\beta,\alpha]} + 4\varrho v_\alpha U_{,t} + 4U_{,\beta} (\varrho v_\alpha v_\beta + p \delta_{\alpha\beta}) = \\
 & = [\varepsilon^1 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5]
 \end{aligned}$$

and taking into account (A.2) we have

$$\begin{aligned}
 & \partial_t \left[\varrho \left(1 + v^2 + 2U + \pi + \frac{p}{\varrho} \right) v_\alpha + 4\varrho \left(v_\alpha U - W_\alpha \right) \right] + \\
 & + \partial_\beta \left[\varrho \left(1 + v^2 + 2U + \pi + \frac{p}{\varrho} \right) v_\alpha v_\beta + p(1+2U) \delta_{\alpha\beta} + 4\varrho v_\beta (v_\alpha U - W_\alpha) \right] - \\
 & - \varrho U_{,\alpha} + 2\varrho U^2_{,\alpha} - \frac{\varrho}{8\pi} I_{-1,\alpha} [00 + \mu\mu] - \\
 & - 2\varrho \varphi U_{,\alpha} + 4\varrho v_\beta W_{\beta,\alpha} = \\
 & = [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5]
 \end{aligned} \tag{A.3}$$

Recalling the definition (2.7) of ϱ^* and that (see ch.1)

$$\begin{aligned}
 & 2\varrho U^2_{,\alpha} - \frac{\varrho}{8\pi} I_{-1,\alpha} [00 + \mu\mu] = \\
 & = -2\varrho \Phi_{,\alpha} + \frac{1}{2} \varrho \chi_{,\alpha} + [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5]
 \end{aligned}$$

we can write (A.3) in the form

$$\begin{aligned}
 & \partial_t \left[\varrho^* \left(1 + \frac{v^2}{2} - U + \pi + \frac{p}{\varrho^*} \right) v_\alpha + 4\varrho^* \left(v_\alpha U - W_\alpha \right) \right] + \\
 & + \partial_\beta \left[\varrho^* \left(1 + \frac{v^2}{2} - U + \pi + \frac{p}{\varrho^*} \right) v_\alpha v_\beta + p(1+2U) \delta_{\alpha\beta} + 4\varrho^* v_\beta \cdot \right. \\
 & \cdot (v_\alpha U - W_\alpha) \left. \right] - \varrho U_{,\alpha} - 2\varrho (\varphi U_{,\alpha} + \Phi_{,\alpha}) + \\
 & + 4\varrho v_\beta W_{\beta,\alpha} + \frac{1}{2} \varrho (W_\alpha - U_{\beta,\alpha\beta}) \varepsilon \\
 & = [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5].
 \end{aligned} \tag{A.4}$$

The time-derivative in (A.4) is now absorbed by the first term, using the fact that

$$\begin{aligned}
& \mathcal{G}(W_\alpha - U_{\beta \cdot \alpha \beta})_{,t} = \partial_t \left[\mathcal{G}(W_\alpha - U_{\beta \cdot \alpha \beta}) \right] - \\
& - \mathcal{G}_{,t} (W_\alpha - U_{\beta \cdot \alpha \beta}) = \\
& = \partial_t \left[\mathcal{G}(W_\alpha - U_{\beta \cdot \alpha \beta}) \right] + (\mathcal{G} v_\mu)_{,\mu} (W_\alpha - U_{\beta \cdot \alpha \beta}) + \\
& + [\mathcal{E}^6 \delta, \mathcal{E}^4 \delta^3, \mathcal{E}^2 \delta^5] = \\
& = \partial_t \left[\mathcal{G}(W_\alpha - U_{\beta \cdot \alpha \beta}) \right] + \partial_\mu \left[\mathcal{G} v_\mu (W_\alpha - U_{\beta \cdot \alpha \beta}) \right] - \\
& - \mathcal{G} v_\mu \partial_\mu (W_\alpha - U_{\beta \cdot \alpha \beta}) + [\mathcal{E}^6 \delta, \mathcal{E}^4 \delta^3, \mathcal{E}^2 \delta^5]
\end{aligned}$$

(A.4) assumes then the form

$$\begin{aligned}
& \partial_t \left[\mathcal{G}^* \left(1 + \frac{v^2}{2} - U + \Pi + \frac{P}{\mathcal{G}^*} \right) v_\alpha + 4 \mathcal{G}^* (v_\alpha U - W_\alpha) + \right. \\
& + \frac{1}{2} \mathcal{G}^* (W_\alpha - U_{\beta \cdot \alpha \beta}) \left. \right] + \partial_\beta \left[\mathcal{G}^* \left(1 + \frac{v^2}{2} - U + \Pi + \frac{P}{\mathcal{G}^*} \right) v_\alpha v_\beta + \right. \\
& + P(1 + 2U) \delta_{\alpha\beta} + 4 \mathcal{G}^* v_\beta (v_\alpha U - W_\alpha) + \frac{1}{2} \mathcal{G}^* v_\beta (W_\alpha - \\
& - U_{\mu \cdot \alpha \mu}) \left. \right] - \mathcal{G} U_{,\alpha} - 2 \mathcal{G}^* (\varphi U_{,\alpha} + \Phi_{,\alpha}) + \\
& + 4 \mathcal{G}^* v_\beta W_{\beta,\alpha} - \frac{1}{2} \mathcal{G}^* v_\beta \partial_\beta (W_\alpha - U_{\mu \cdot \alpha \mu}) = \\
& = [\mathcal{E}^6 \delta, \mathcal{E}^4 \delta^3, \mathcal{E}^2 \delta^5]
\end{aligned}$$

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Chapter IV

Relativistic Gravitational Interactions between Bodies
of Finite Dimensions (Part II)

Abstract: An equation of motion for a perfect fluid body in a N-body system is obtained within the framework of the PPN approximation to fully conservative theories of gravity, along the same lines as in the previous approach (ch.3), valid for General Relativity. For a bound system composed of two rigidly rotating spheroids, the Lagrangian is constructed and the properties of the orbital and rotational motion of a given body are analysed.

We generalize some theorems on the relativistic motion of extended bodies, which so far had been proved only under very restrictive assumptions on the nature both of the bodies and of the motion itself.

We emphasize that we use a hydrodynamical description of the bodies, i.e. we consider extended, non test-bodies having internal structure and also having a self-field, which is essential in the definition of the mass, though the self-force does not affect the overall equation of motion of the bodies.

(1): Introduction

In the previous chapter an approach was discussed within a first post-newtonian approximation to General Relativity, which yielded a formulation of an equation of motion for an extended fluid body in an isolated N-body system.

Under the assumption that each body of the system is spherically symmetric and in pure translational motion, the equation of motion was shown to be derivable from a Lagrangian having the same structure as the Einstein-Infeld-Hoffmann(EIH) Lagrangian. Originally this was obtained by a completely different method, in which the bodies were treated as field singularities.

In this chapter, the equation of motion for an extended, perfect fluid body derived in chapter 3 is generalized by the inclusion of the PPN parameters β and γ (sect.2). In section 3 each body is assumed to be a spheroid in rigid rotation, and it is shown that an equation of motion can then be derived from a Lagrangian which generalizes to rigidly rotating spheroids the Lagrangian derived in chapter 3, valid for spherically symmetric bodies in pure translational motion. In section 4 the motion of a relativistic two body system is investigated. In subsect. 4a the location of the center of mass of the whole system is discussed. In subsect. 4b, using the equation of perturbation of the orbital elements, secular, relativistic perturbations of the longitude of the ascending node, the periastron and the mean anomaly of the epoch are derived.

A generalization of the Lense-Thirring effects is obtained, allowing for rotation of both the bodies. In subsection 4c we propose a definition of the spin consistent to the post-newtonian order of accuracy, and we study the rotational motion of a given body of the system: using the equation of conservation of the total angular momentum of the system, it is shown that during the orbital motion, the spin precesses about the direction of the orbital angular momentum. In sect. 5 the main results obtained in this chapter are discussed, with a numerical application to the most recent observations of the binary pulsar.

(2): Parametrized Post-newtonian Hydrodynamics

In this section, the spatial part of the local hydrodynamical equations

$$T^{ab}_{;b} = 0 \quad (2.1)$$

will be written in the post-newtonian approximation to "fully conservative theories of gravity": this means that the equation will differ from the one written in ch. 3 by the presence of two parameters, β and γ , which differ from one in other theories of gravity (like e.g. the scalar-tensor theory of Brans-Dicke-Jordan).

"Fully conservative theories" admit a set of ten first integrals.⁽⁺⁾ With the use of the equations of motion in the newtonian approximation (continuity equation, Euler equation), the parametrized-post-newtonian (PPN) version of (2.1)_{a= α} will be reformulated in the following way

$$\partial_t \pi_\alpha + \partial_\beta \sigma_{\alpha\beta} = f_\alpha \quad (2.2)$$

by analogy with the Euler equation, along the same lines as in chapter 3. By integration of (2.2) over the volume of an extended body, an equation of motion will follow.

Of the post-newtonian potentials introduced in chapter 3, only the non linear one needs to be modified because of the

⁽⁺⁾Detailed discussion on this subject may be found in [15].

PPN parameters. According to [15], the effective density for the non linear potential $\bar{\Phi}$ has the form

$$\varrho \varphi := \varrho \left(\frac{1+\gamma}{2} v^2 + \frac{1+3\gamma-2\beta}{2} U + \gamma \frac{3p}{2\varrho} + \frac{\pi}{2} \right) \quad (2.3)$$

and the associated potential is

$$8\pi \bar{\Phi} = \int \varrho' \varphi' |x-x'|^{-1} d^3x'.$$

The notations indicating the "self" and "external" parts of the potentials are the same as in chapter 3.

In the PPN framework the density satisfying the three dimensional form of the law of conservation of the baryonic number is

$$\varrho^* = \varrho \left(1 + \frac{v^2}{2} + 3\gamma U \right) + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]. \quad (2.4)$$

The PPN expressions for the components of the energy-momentum tensor are

$$\begin{aligned} T^{00} &= \varrho \left(1 + \pi + v^2 + 2U \right) + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \\ T^{0\alpha} &= T^{00} v_\alpha + p v_\alpha + [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \\ T^{\alpha\beta} &= T^{00} v_\beta + p (1 - 2\gamma U) \delta_{\alpha\beta} + [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5], \end{aligned} \quad (2.5)$$

while the Christoffel symbols needed in the equation of motion are

$$\left\{ \begin{matrix} \alpha \\ 00 \end{matrix} \right\} = -U_{,\alpha} + [(\beta + \gamma) U^2 - 2\Phi]_{,\alpha} - \frac{4\gamma+3}{2} W_{\alpha,t} - \frac{1}{2} U_{\beta\alpha\beta,t} + [\varepsilon^4 \delta, \varepsilon^2 \delta^3]$$

$$\left\{ \begin{matrix} \alpha \\ \beta 0 \end{matrix} \right\} = \gamma U_{,t} - (2\gamma + 2) W_{[\alpha, \beta]} + [\varepsilon^4, \varepsilon^2 \delta^2] \quad (2.6)$$

$$\left\{ \begin{matrix} \alpha \\ \beta \gamma \end{matrix} \right\} = \gamma (U_{,\beta} \delta_{\alpha\gamma} + U_{,\gamma} \delta_{\alpha\beta} - U_{,\alpha} \delta_{\beta\gamma}) + [\varepsilon^2 \delta]$$

The expressions (2.5) and (2.6) are all we need in order to write the spatial part of the local equations of motion (2.1).

The result is

$$\begin{aligned} & \partial_t \left[\rho (1 + v^2 + \pi + 2U + \frac{p}{\rho}) v_\alpha \right] + \partial_\beta \left[\rho (1 + v^2 + 2U + \pi + p/\rho) v_\alpha v_\beta + \right. \\ & \left. + p(1 - 2\gamma U) \delta_{\alpha\beta} \right] + \rho \left[-U_{,\alpha} + (\beta + \gamma) U^2_{,\alpha} - 2\Phi_{,\alpha} - \right. \\ & \left. - \frac{4\gamma+3}{2} W_{\alpha,t} - \frac{1}{2} U_{\beta\alpha\beta,t} \right] - \rho (v^2 + 2U + \pi) U_{,\alpha} + \\ & + 2\rho v_\beta \left[\gamma U_{,t} \delta_{\alpha\beta} - (2\gamma + 2) W_{[\alpha, \beta]} \right] + (3\gamma - 1) \rho v_\alpha U_{,t} + \\ & + \gamma (U_{,\gamma} \delta_{\alpha\beta} + U_{,\beta} \delta_{\alpha\gamma} - U_{,\alpha} \delta_{\beta\gamma}) \quad (2.7) \end{aligned}$$

$$\begin{aligned} & \cdot (\rho v_\beta v_\gamma + p \delta_{\beta\gamma}) + (3\gamma - 1) U_{,\beta} (\rho v_\alpha v_\beta + \\ & + p \delta_{\alpha\beta}) = [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \end{aligned}$$

Using, in the post-newtonian terms in (2.7), the fact that, at the lowest order

$$\partial_t \varrho + \partial_\mu (\varrho v_\mu) = [\varepsilon^4, \varepsilon^2 \delta^2] \quad (2.8)$$

$$\begin{aligned} \partial_t (\varrho v_\alpha) + \partial_\beta (\varrho v_\alpha v_\beta + p \delta_{\alpha\beta}) = \\ = \varrho U_{,\alpha} + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad , \end{aligned} \quad (2.9)$$

and working along the same lines as in chapter 3 (appendix), one obtains, with reference to eq. (2.2), that (2.2) holds if we take

$$\begin{aligned} \pi_\alpha := \varrho^* \left(1 + \frac{v^2}{2} - U + \pi + p/\varrho^* \right) v_\alpha + \frac{1}{2} \varrho^* \left(W_\alpha - \right. \\ \left. - U_{\beta \cdot \alpha \beta} \right) + (2\gamma + 2) \varrho^* (v_\alpha U - W_\alpha) \end{aligned} \quad (2.10)$$

$$\sigma_{\alpha\beta} := \pi_\alpha v_\beta + [1 + (3\gamma - 1)U] p \delta_{\alpha\beta} \quad (2.11)$$

$$\begin{aligned} f_\alpha := \varrho U_{,\alpha} + 2 \varrho^* (\varphi U_{,\alpha} + \Phi_{,\alpha}) - \\ - (2 + 2\gamma) \varrho^* v_\nu \partial_\alpha W_\nu + \\ + \frac{1}{2} \varrho^* v_\beta \partial_\beta (W_\alpha - U_{\gamma \cdot \alpha \gamma}) \quad . \end{aligned} \quad (2.12)$$

Everywhere in this expression the density ϱ has been replaced by the conserved density ϱ^* introduced in eq.(2.4). This in particular holds for the first element of (2.12), where the density ϱ is now considered as a functional of ϱ^* , $\underline{v}$ and U ,

$$\varrho = \varrho^* \left(1 - \frac{v^2}{2} - 3 \varrho U \right) + [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (2.13)$$

by inversion of (2.4).

Since the quantities π_α , $\sigma_{\alpha\beta}$, f_α enter an equation similar to the Euler equation

$$\partial_t \pi_\alpha + \partial_\beta \sigma_{\alpha\beta} = f_\alpha, \quad (2.14)$$

they can be named respectively PPN momentum density, stress density and force density.

Proceeding in analogy to newtonian theory, one can integrate (2.14) over the volume of the generic body A of the system, obtaining

$$\frac{d}{dt} \int_A \pi_\alpha d^3x = \int_A f_\alpha d^3x. \quad (2.15)$$

Clearly the integral of the divergence term does not contribute, since the integrand has compact support.

Defining

$$P_\alpha := \int_A \pi_\alpha d^3x \quad ; \quad F_\alpha := \int_A f_\alpha d^3x$$

eq. (2.15) is rewritten as follows

$$\frac{d P_\alpha}{dt} = F_\alpha \quad (2.16)$$

A splitting of the velocity-field inside the body A is now introduced. Calling $\underline{a}(t)$ the vector defining the location of a -not yet specified- reference point within A, $\dot{\underline{a}}$ its velocity and $\dot{\xi}(\underline{x}, t)$ the velocity, with respect to the reference point, of the element of A occupying the position $\underline{x}$ at the time t, we write

$$v_{\alpha}(\underline{x}, t) = \dot{a}_{\alpha}(t) + \dot{\xi}_{\alpha}(\underline{x}, t)$$

Define

$$8\pi \bar{w}_{(a)}^{\alpha} := \int_A \varrho' \dot{\xi}'_{\alpha} |\underline{x} - \underline{x}'|^{-1} d^3x'$$

$$8\pi \mu_{(a)}^{\alpha\beta} := \int_A \varrho' \frac{(\underline{x}_{\alpha} - \underline{x}'_{\alpha})(\underline{x}_{\beta} - \underline{x}'_{\beta})}{|\underline{x} - \underline{x}'|^3} d^3x' \quad (2.17)$$

$$8\pi \bar{\mu}_{(a)}^{\beta \cdot \alpha\beta} := \int_A \varrho' \dot{\xi}'_{\beta} \frac{(\underline{x}_{\alpha} - \underline{x}'_{\alpha})(\underline{x}_{\beta} - \underline{x}'_{\beta})}{|\underline{x} - \underline{x}'|^3} d^3x'$$

$$D_{\alpha} := \int_A \varrho^* \left\{ \left(1 + \frac{\dot{\xi}^2}{2} - \mu_a + \pi + p/\varrho^* \right) \dot{\xi}_{\alpha} + \frac{1}{2} (\bar{w}_{(a)}^{\alpha} - \right.$$

$$\left. - \bar{\mu}_{(a)}^{\beta \cdot \alpha\beta} \right) - (2\gamma + 2) (\bar{w}_{(a)}^{\alpha} - \dot{\xi}_{\alpha} \mu_a) \right\} d^3x \quad (2.18)$$

$$I_A^{\alpha\beta} := \int_A \varrho^* \xi_{\alpha} \xi_{\beta} d^3x \quad (2.19)$$

$$m_A := \int_A \varrho^* \left(1 + \frac{\dot{\xi}^2}{2} - \frac{\mu_a}{2} + \pi \right) d^3x \quad (2.20)$$

where D_α is the generalization of the newtonian dipole moment, $I_{\alpha\beta}$ is the moment of inertia and m_A is the inertial mass, the properties of which have been discussed in chapter 2.

By means of the newtonian tensor-virial equation

$$\frac{1}{2} \ddot{I}_{\alpha\beta} = \int_A (\varrho^* \dot{\xi}_\alpha \dot{\xi}_\beta - \frac{1}{2} W_{\alpha\beta} \varrho^* + P \delta_{\alpha\beta}) d^3x, \quad (2.21)$$

the momentum P_α is decomposed as follows

$$\begin{aligned} P_\alpha = & \left(1 + \frac{\dot{a}^2}{2}\right) m_A \dot{a}_\alpha + \left(1 + \frac{\dot{a}^2}{2}\right) D_\alpha + \\ & + \frac{1}{2} \dot{a}_\beta \ddot{I}_{\alpha\beta} + \dot{a}_\alpha (\underline{\dot{a}} \cdot \underline{D}) - \\ & - \int_A \varrho^* \left[\frac{4\gamma+3}{2} W_\alpha^{(a)} + \frac{1}{2} U_{\beta,\alpha\beta}^{(a)} - \right. \\ & \left. - (2\gamma+1) (\dot{a}_\alpha + \dot{\xi}_\alpha) U^{(a)} \right] d^3x, \end{aligned} \quad (2.22)$$

Also the newtonian force F_α can be similarly decomposed, but the result does not appear illuminating.

In the next section, additional assumptions on the system will be introduced. Although limiting somewhat the generality of the equations, these assumptions are needed in order to model the bodies and make quantitative statements.

(3): The two body problem

In this section a bound system composed of two bodies, A and B, will be considered. Each body will hereafter be assumed to fulfill the following requirements:

- a): in each body there is a point (necessarily unique) such that the density ρ^* and the internal velocity $\dot{\xi}$ are invariant under rotations about an axis through the point, and under reflections in a plane orthogonal to the axis;
- b): each body is in rigid rotation about its symmetry axis;
- c): for each body the ratio (radius)/(mutual distance) is small compared to unity.

Assumption a) means that an observer at the particular point of a body, the existence of which is a priori assumed, and using a local coordinate system $(\xi_\alpha, \tilde{t})$ ⁽⁺⁾ with origin at that point, will recognize the body to be an ellipsoid of rotation, having in fact axial and "specular" symmetry.

⁽⁺⁾ Such coordinates $(\tilde{\xi}_\alpha, \tilde{t})$ can be obtained as follows: write the approximate metric in terms of one-forms

$$ds^2 = -(\omega^0)^2 + (\omega^1)^2 + (\omega^2)^2 + (\omega^3)^2$$

and assume that functions $\bar{x}^a$ exist, such that $d\bar{x}^a = \omega^a$ is true consistently with the order of approximation. Integration of this system gives the approximately inertial, local coordinates $\bar{x}^a$, and their Lorentz transformation to the center of the body A gives the $(\tilde{\xi}_\alpha, \tilde{t})$.

Assumption a) fixes uniquely inside each body the reference point, which in the previous section was left unspecified.

Assumption b) is here supposed to mean that, with respect to the local comoving coordinate system $(\tilde{\xi}_\alpha, \tilde{t})$, one has

$$\ddot{\tilde{\xi}}_\alpha := \frac{d\dot{\tilde{\xi}}_\alpha}{d\tilde{t}} = \omega_{\alpha\beta}(\tilde{t}) \tilde{\xi}_\beta, \quad (3.1)$$

with $\omega_{\alpha\beta} = \omega[\alpha\beta]$.

The aim of this section is to prove that if assumptions a), b), c) hold, then there exists a Lagrangian from which (2.16) can be derived as Euler-Lagrange equation.

To this purpose the following theorem will be needed:

Theorem: if a) and b) hold, then the quantity D_α defined in (2.18) is of order

$$D_\alpha = [\varepsilon', \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (3.2)$$

The proof is given in the appendix.

We notice that in the post-newtonian equations of motion the omitted terms are of order $[\varepsilon' \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5]$.

The theorem then implies that in the time-derivative of $\underline{P}$ (see (2.22)), $\dot{\underline{D}}$ can be substituted by zero.

In chapter 2 it was proved that if the body A (say) moves rigidly, then the inertial mass defined by (2.20) is approximately independent of time. In fact its time-evolution was shown to be described by the time variation of the traceless part of the mass-multipole moments. In the present

case m_A can then be considered as a constant, since each body keeps its shape and no deformation is assumed to occur. This also means that the time-variation of the tensor of inertia $\bar{I}_{\alpha\beta}$ will be at most periodic and its average value will be zero on a time-scale large compared to rotation period.

Equation (2.22) can then be rewritten in the following way:

$$P_\alpha = \left(1 + \frac{\dot{a}^2}{2}\right) m_A \dot{a}_\alpha - \int_A \rho^* \left[\frac{4\gamma+3}{2} W_\alpha^{(a)} + \frac{1}{2} U_{\beta \cdot \alpha \beta}^{(a)} - (2\gamma+1)(\dot{a}_\alpha + \dot{\xi}_\alpha) U^{(a)} \right] d^3x. \quad (3.3)$$

Assumption c) can now be used as follows: consider the expansion for the inverse of the distance

$$|x - x'|^{-1} = 1/R + \xi_\alpha \frac{\partial}{\partial a_\alpha} \frac{1}{R} + \xi'_\beta \frac{\partial}{\partial b_\beta} \frac{1}{R} + O(R^{-3}), \quad (3.4)$$

where $x \in \text{supp}(A)$, $x' \in \text{supp}(B)$, $\underline{R} := \underline{a} - \underline{b}$, $R := |\underline{R}|$, $\xi'_\alpha = x'_\alpha - b_\alpha$.

Eq. (3.4) can also be rewritten as follows

$$|x - x'|^{-1} = \frac{1}{R} + \xi'_\beta \frac{R_\beta}{R^3} - \xi_\alpha \frac{R_\alpha}{R^3} + O(R^{-3}). \quad (3.5)$$

In view of a similar multipole expansion of the external tensor potential $U_{\beta \cdot \alpha \beta}^{(a)}$, also the following formula has to be considered:

$$\begin{aligned}
\frac{(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|x - x'|^3} &= \frac{R_\alpha R_\beta}{R^3} + \xi_\gamma \frac{\delta_{\alpha\gamma} R_\beta + \delta_{\beta\gamma} R_\alpha}{R^3} - \\
&- 3 \xi_\gamma \frac{R_\alpha R_\beta R_\gamma}{R^5} - \xi'_\gamma \frac{\delta_{\alpha\gamma} R_\beta + \delta_{\beta\gamma} R_\alpha}{R^3} + \\
&+ 3 \xi'_\gamma \frac{R_\alpha R_\beta R_\gamma}{R^5} + O(R^{-3}) .
\end{aligned} \quad (3.6)$$

Before inserting these formulas into the external potentials in (3.3), it is important to note that the tensor

$$S_{\alpha\beta} = 2 \int_A \mathcal{S}^* \xi_\alpha \dot{\xi}_\beta d^3x$$

(likewise S_B for the body B) is antisymmetric, since its symmetric part is proportional to the time derivative of the tensor of inertia, $\dot{I}_A$, assumed to be zero on average.

Beside the $S_{\alpha\beta}$ tensor (spin-tensor), also its dual will be used:

$$S_{\alpha\beta} = \varepsilon_{\alpha\beta\gamma} S_\gamma \quad ; \quad S_\gamma = \frac{1}{2} \varepsilon_{\alpha\beta\gamma} S_{\alpha\beta}$$

where, as usual, $\varepsilon_{\alpha\beta\gamma}$ is the completely antisymmetric tensor:

$$\varepsilon_{\alpha\beta\gamma} = \begin{cases} 1 & (\alpha\beta\gamma) \text{ even permutation} \\ -1 & (\alpha\beta\gamma) \text{ odd permutation} \\ 0 & \text{otherwise} \end{cases}$$

We have then in (3.3)

$$\begin{aligned} 8\pi \int_A \varrho^* W_\alpha^{(a)} d^3x &= \int_A \varrho^* d^3x \int_B \varrho^{*'} v'_\alpha |x-x'|^{-1} d^3x' = \\ &= \frac{m_A m_B}{R} \dot{b}_\alpha - \frac{m_A}{2} \left(\frac{\underline{R} \times \underline{S}_B}{R^3} \right)_\alpha + O(R^{-3}) \end{aligned} \quad (3.7)$$

by virtue of the expression (3.5), of the splitting of the velocity $\underline{v}'$ within B in the sum of the velocity $\dot{\underline{b}}$ of the reference point and of the velocity (of rigid rotation) $\underline{\xi}$ relative to it, and of the symmetry requirement b), by which the dipole moment is zero.

Likewise for the tensor potential, using (3.6)

$$\begin{aligned} 8\pi \int_A \varrho^* U_{\beta \cdot \alpha \beta}^{(a)} d^3x &= \int_A \varrho^* d^3x \int_B \varrho^{*'} v'_\beta \frac{(x_\alpha - x'_\alpha)(x_\beta - x'_\beta)}{|x-x'|^3} d^3x' \\ &= \frac{m_A m_B}{R^3} (\dot{\underline{b}} \cdot \underline{R}) R_\alpha - \frac{m_A}{2} \left(\frac{\underline{R} \times \underline{S}_B}{R^3} \right)_\alpha + \\ &+ O(R^{-3}) \end{aligned} \quad (3.8)$$

Finally the part of (3.3) containing the scalar potential is reduced to the form

$$\begin{aligned} 8\pi \int_A \varrho^* (\dot{a}_\alpha + \xi_\alpha) U^{(a)} d^3x &= \frac{m_A m_B}{R} \dot{a}_\alpha + \\ &+ \frac{m_B}{2 R^3} (\underline{R} \times \underline{S}_A)_\alpha + O(R^{-3}) \end{aligned} \quad (3.9)$$

Equation (3.3) can be rewritten as follows

$$\begin{aligned}
 P_\alpha = & \left(1 + \frac{\dot{\underline{a}}^2}{2}\right) m_A \dot{a}_\alpha + (2\gamma+1) \frac{m_A m_B}{8\pi R} \dot{a}_\alpha - \frac{4\gamma+3}{2} \frac{m_A m_B}{8\pi R} \dot{b}_\alpha \\
 & - \frac{1}{2} \frac{m_A m_B}{8\pi R^3} (\dot{\underline{b}} \cdot \underline{R}) R_\alpha + \frac{2\gamma+1}{2} \frac{m_B}{8\pi R^3} (\underline{R} \times \underline{S}_A)_\alpha + \\
 & + (1+\gamma) \frac{m_A}{8\pi R^3} (\underline{R} \times \underline{S}_B)_\alpha + O(R^{-3})
 \end{aligned} \tag{3.10}$$

or equivalently

$$\begin{aligned}
 P_\alpha = & \frac{\partial}{\partial \dot{a}_\alpha} \left\{ \frac{m_A}{2} \left(1 + \frac{\dot{\underline{a}}^2}{4}\right) \dot{a}^2 + \frac{m_B}{2} \left(1 + \frac{\dot{\underline{b}}^2}{4}\right) \dot{b}^2 + \right. \\
 & + \frac{m_A m_B}{8\pi R} \left[\frac{2\gamma+1}{2} (\dot{a}^2 + \dot{b}^2) - \frac{4\gamma+3}{2} \dot{\underline{a}} \cdot \dot{\underline{b}} - \frac{1}{2} \frac{(\dot{\underline{a}} \cdot \underline{R})(\dot{\underline{b}} \cdot \underline{R})}{R^2} \right] \\
 & + \frac{2\gamma+1}{2} \left[\frac{m_B}{8\pi R^3} (\dot{\underline{a}} \times \underline{R}) \cdot \underline{S}_A - \frac{m_A}{8\pi R^3} (\dot{\underline{b}} \times \underline{R}) \cdot \underline{S}_B \right] - \\
 & - (1+\gamma) \left[\frac{m_B}{8\pi R^3} (\dot{\underline{b}} \times \underline{R}) \cdot \underline{S}_A - \frac{m_A}{8\pi R^3} (\dot{\underline{a}} \times \underline{R}) \cdot \underline{S}_B \right] \left. \right\} + \\
 & + O(R^{-3})
 \end{aligned} \tag{3.11}$$

where the arbitrary constants (with respect to $\dot{\underline{a}}$) of integration have been chosen in such a way that the whole expression is symmetric in the exchange of the dynamical

variables of the bodies.

In a similar way, the reduction of the post-newtonian force is considered. As in chapter 3, this force is defined as the integral of (2.12) over the volume element of the body A.

We consider only the "external force", since -in the same way as in ch. 3- it can be proved that the self-force gives no contribution to the equation of motion (2.16):

$$\begin{aligned}
 F_{\alpha}^{(ext)} := & \int_A \rho U_{,\alpha}^{(a)} d^3x + 2 \int_A \rho^* (\varphi U_{,\alpha}^{(a)} + \Phi_{,\alpha}^{(ext)}) d^3x - \\
 & - (2\gamma + 2) \int_A \rho^* v_{\nu} \partial_{\alpha} W_{\nu}^{(a)} d^3x + \\
 & + \frac{1}{2} \int_A \rho^* v_{\beta} \partial_{\beta} (W_{\alpha}^{(a)} - U_{\beta, \alpha\beta}^{(a)}) d^3x
 \end{aligned} \tag{3.12}$$

The newtonian term is first reduced. Using (2.4) we obtain

$$\begin{aligned}
 \int_A \rho \partial_{\alpha} U^{(a)} d^3x &= \int_A d^3x \rho^* \left(1 - \frac{v^2}{2} - 3\gamma U\right) \cdot \\
 \cdot \int_B d^3x' \rho^{*'} \left(1 - \frac{v'^2}{2} - 3\gamma U'\right) \partial_{\alpha} |x - x'|^{-1}
 \end{aligned} \tag{3.13}$$

Introduce the notations

$$\begin{aligned}
 m_A^* &:= \int_A \rho^* d^3x ; \quad \varepsilon_A := \frac{1}{2} \int_A \rho^* u_a d^3x \\
 T_A &:= \frac{1}{2} \int_A \rho^* \dot{\xi}^2 d^3x
 \end{aligned} \tag{3.14}$$

(likewise for B; when the integral is extended to B, $\dot{\xi}$ is intended to be $\dot{\xi} = \underline{v} - \dot{\underline{b}}$), so that from (2.20) it follows

$$m_A = m_A^* + T_A - \varepsilon_A \quad (3.15)$$

Then (3.13) becomes, via (3.4) and the usual splitting of the potentials into self and external part

$$\begin{aligned} 8\pi \int_A \varrho \partial_\alpha U^{(a)} d^3x &= \left[m_A^* m_B^* \left(1 - \frac{\dot{a}^2}{2} - \frac{\dot{b}^2}{2} \right) - \right. \\ &- 6\gamma (\varepsilon_A m_B^* + \varepsilon_B m_A^*) - T_A m_B^* - T_B m_A^* \left. \right] \frac{\partial}{\partial a_\alpha} \frac{1}{R} - \\ &- \frac{3}{2} \gamma \frac{m_A m_B}{8\pi} (m_A + m_B) \frac{\partial}{\partial a_\alpha} \frac{1}{R^2} + \\ &+ \frac{\partial}{\partial a_\alpha} \left[\frac{m_B}{2} \frac{(\underline{R} \times \underline{\dot{a}})}{R^3} \cdot \underline{S}_A + \frac{m_A}{2} \frac{(\underline{R} \times \underline{\dot{b}})}{R^3} \cdot \underline{S}_B \right] \quad (3.16) \\ &+ O(R^{-4}) \end{aligned}$$

Next consider in (3.12)

$$\begin{aligned} 2 \int_A \varrho^* \varphi \partial_\alpha U^{(a)} d^3x &= \int_A d^3x \varrho^* \left(\frac{1+\gamma}{2} v^2 + \frac{3\gamma-2\beta+1}{2} U + \right. \\ &\left. + \frac{3\gamma p}{2\varrho^*} \right) \int_B d^3x' \varrho^{*'} \partial_\alpha |\underline{x} - \underline{x}'|^{-1} \end{aligned}$$

which, making use of the usual multipole expansion and of the virial theorem (in order to eliminate the integrated pressure),

becomes

$$\begin{aligned}
 8\pi \cdot 2 \int_A g^* \varphi \partial_\alpha U^{(\alpha)} d^3x &= \left[(1+\gamma) m_A m_B \dot{a}^2 + 2 m_B T_A + \right. \\
 &+ 2 \left(\frac{7}{2} \gamma - 2\beta + 1 \right) m_B \epsilon_A \left. \right] \frac{\partial}{\partial a_\alpha} \frac{1}{R} + (1+3\gamma-2\beta) \frac{m_A m_B^2}{8\pi R} \frac{\partial}{\partial a_\alpha} \frac{1}{R^2} \\
 &- \frac{\partial}{\partial a_\alpha} \left[2(1+\gamma) m_B \dot{a}_\mu \frac{R_\Lambda}{R^3} \int_A g^* \dot{\xi}_\mu \xi_\Lambda d^3x \right] + \\
 &+ O(R^{-4})
 \end{aligned}$$

and, using the definition of the spin and (3.5),

$$\begin{aligned}
 8\pi \cdot 2 \int_A g^* \varphi \partial_\alpha U^{(\alpha)} d^3x &= \frac{\partial}{\partial a_\alpha} \left\{ (1+\gamma) \frac{m_A m_B}{R} \dot{a}^2 + \right. \\
 &+ 2 \frac{m_B}{R} \left[T_A + \left(\frac{7}{2} \gamma - 2\beta + 1 \right) \epsilon_A \right] + \frac{1+3\gamma-2\beta}{2} \frac{m_A m_B^2}{8\pi R^2} \\
 &- (1+\gamma) \frac{m_B}{R^3} (\underline{R} + \underline{\dot{a}}) \cdot \underline{S}_A \left. \right\} + \quad (3.17) \\
 &+ O(R^{-4})
 \end{aligned}$$

The twin term in (3.12)

$$\int_A g^* \partial_\alpha \Phi^{(\alpha)} d^3x$$

differs from (3.17) just by interchange of A with B;

$$\begin{aligned}
8\pi \cdot 2 \int_A \mathcal{G}^* \partial_\alpha \Phi^{(ext)} d^3x &= \frac{\partial}{\partial a_\alpha} \left\{ (1+\gamma) \frac{m_A m_B}{R} \dot{b}^2 + \right. \\
&+ 2 \frac{m_A}{R} \left[T_B + \left(\frac{7}{2} \gamma - 2\beta + 1 \right) \varepsilon_B \right] + \frac{1+3\gamma-2\beta}{2} \frac{m_A^2 m_B}{8\pi R^2} \\
&+ \left. (1+\gamma) m_A \frac{(\underline{R} \times \underline{\dot{b}})}{R^3} \cdot \underline{S}_B \right\} + \\
&+ O(R^{-4}), \tag{3.18}
\end{aligned}$$

so that, collecting (3.17) and (3.18)

$$\begin{aligned}
8\pi \cdot 2 \int_A \mathcal{G}^* \left(\varphi \partial_\alpha U^{(a)} + \partial_\alpha \Phi^{(ext)} \right) d^3x &= \\
&= \frac{\partial}{\partial a_\alpha} \left\{ (1+\gamma) \frac{m_A m_B}{8\pi R} (\dot{a}^2 + \dot{b}^2) + 2 \frac{m_B}{R} \left[T_A + \right. \right. \\
&+ \left. \left. \left(\frac{7}{2} \gamma - 2\beta + 1 \right) \varepsilon_A \right] + \frac{2m_A}{R} \left[T_B + \left(\frac{7}{2} \gamma - 2\beta + 1 \right) \varepsilon_B \right] \right. \tag{3.19} \\
&+ \frac{1+3\gamma-2\beta}{2} \frac{m_A m_B}{8\pi R^2} (m_A + m_B) - (1+\gamma) m_B \frac{(\underline{R} \times \underline{\dot{a}})}{R^3} \cdot \underline{S}_A + \\
&+ \left. (1+\gamma) m_A \frac{(\underline{R} \times \underline{\dot{b}})}{R^3} \cdot \underline{S}_B \right\} + O(R^{-4}).
\end{aligned}$$

Considering the next term in (3.12), one has

$$\begin{aligned}
-8\pi (2\gamma+2) \int_A \mathcal{G}^* \sqrt{\nu} \partial_\alpha W_\nu^{(a)} d^3x &= \\
&= \frac{\partial}{\partial a_\alpha} \left[- (2\gamma+2) \frac{m_A m_B}{R} \underline{\dot{a}} \cdot \underline{\dot{b}} + (1+\gamma) \cdot \right. \tag{3.20} \\
&\quad \left. m_B \frac{(\underline{R} \times \underline{\dot{b}})}{R^3} \cdot \underline{S}_A - (1+\gamma) m_A \frac{(\underline{R} \times \underline{\dot{a}})}{R^3} \cdot \underline{S}_B \right] + O(R^{-4}).
\end{aligned}$$

Finally there remains:

$$\frac{1}{2} \int_A g^* v_\mu \partial_\mu (W_\alpha^{(a)} - U_{\beta \cdot \alpha \beta}^{(a)}) d^3x$$

which is easily seen to be free of contribution from the spin.

In fact

$$\begin{aligned} & \frac{1}{2} \int_A g^* v_\mu \partial_\mu (W_\alpha^{(a)} - U_{\beta \cdot \alpha \beta}^{(a)}) d^3x = \\ & = \frac{1}{2} \int_A g^* v_\mu \partial_\alpha (W_\mu^{(a)} - U_{\beta \cdot \mu \beta}^{(a)}) d^3x + [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] = \\ & = \frac{1}{2} \int_A g^* v_\mu \left\{ \int_B g^{*'} v'_\beta \partial_{\alpha \beta \mu}^3 |x-x'| d^3x' \right\} d^3x + \\ & \quad + [\varepsilon^6 \delta, \varepsilon^4 \delta^3, \varepsilon^2 \delta^5] \end{aligned}$$

because of the equations of motion at the newtonian order, so that, dropping the error terms and considering the usual multipole expansion for $|x-x'|$ and its derivatives, one obtains up to the dipole term

$$\begin{aligned} & 8\pi \cdot \frac{1}{2} \int_A g^* v_\mu \left\{ \int_B g^{*'} v'_\beta \partial_{\alpha \beta \mu}^3 |x-x'| \right\} d^3x' = \\ & = \frac{1}{2} m_A m_B \ddot{a}_\mu \ddot{b}_\beta \partial_{\alpha \beta \mu}^3 R + \\ & \quad + \frac{1}{2} m_B \ddot{b}_\beta \partial_{\alpha \beta \mu \lambda}^4 R \int_A g^* \dot{\xi}_\mu \dot{\xi}_\lambda d^3x + \\ & \quad + O(R^{-4}) \end{aligned} \tag{3.21}$$

In the second term of equation (3.21) the indices of the anti-symmetric spin tensor are coupled to the symmetric ones of the partial derivatives of R , so that the net contribution is zero. Developing the partial derivative in the first term, one has

$$\begin{aligned} & \frac{1}{2} \cdot 8\pi \int_A g^{\star} v_{\mu} \partial_{\mu} (U_{\alpha}^{(a)} - U_{\beta-\alpha\beta}^{(a)}) d^3x = \\ & = \frac{\partial}{\partial a_{\alpha}} \left\{ \frac{1}{2} \frac{m_A m_B}{R} \left[\underline{\dot{a}} \cdot \underline{\dot{b}} - \frac{(\underline{\dot{a}} \cdot \underline{R})(\underline{\dot{b}} \cdot \underline{R})}{R^2} \right] \right\} (3.22) \\ & + O(R^{-4}) \end{aligned}$$

Collecting the partial results (3.16), (3.19), (3.20) and (3.22), the force (3.12) acting on the body A can finally be rewritten as

$$\begin{aligned} 8\pi F_{\alpha} = & \frac{\partial}{\partial a_{\alpha}} \left\{ \frac{m_A^{(4)} m_B^{(4)}}{R} + \frac{m_A m_B}{R} \left[\frac{1+2\gamma}{2} (\dot{a}^2 + \dot{b}^2) - \right. \right. \\ & \left. \left. - \frac{4\gamma+3}{2} (\underline{\dot{a}} \cdot \underline{\dot{b}}) - \frac{1}{2} \frac{(\underline{\dot{a}} \cdot \underline{R})(\underline{\dot{b}} \cdot \underline{R})}{R^2} \right] + \frac{1-2\beta}{2 \cdot 8\pi R^2} m_A m_B \cdot \right. \\ & \left. \cdot (m_A + m_B) \right\} + \\ & + \frac{\partial}{\partial a_{\alpha}} \left\{ - \frac{1+2\gamma}{2} m_B \frac{\underline{R} \times \underline{\dot{a}}}{R^3} \cdot \underline{S}_A + \frac{1+2\gamma}{2} m_A \frac{\underline{R} \times \underline{\dot{b}}}{R^3} \cdot \underline{S}_B + \right. \\ & \left. + (1+\gamma) m_B \frac{\underline{R} \times \underline{\dot{b}}}{R^3} \cdot \underline{S}_A - (1+\gamma) m_A \frac{\underline{R} \times \underline{\dot{a}}}{R^3} \cdot \underline{S}_B \right\} + \\ & + O(R^{-4}) \end{aligned} \quad (3.23)$$

In (3.23) the (constant) gravitational mass has been introduced, according to the following definition:

$$\begin{aligned}
 m_A^{(g)} &:= m_A^* \left[1 + \frac{T_A}{m_A^*} - \frac{\varepsilon \dot{A}}{m_A^*} - (4\beta - \gamma - 3) \frac{\varepsilon A}{m_A^*} \right] = \\
 &= m_A \left(1 - \eta \frac{\varepsilon A}{m_A} \right)
 \end{aligned} \tag{3.24}$$

(likewise for B), where

$$\eta := 4\beta - \gamma - 3$$

is the Nordtvedt parameter, zero in General Relativity but non zero in other theories of gravity like e.g. Brans-Dicke-Jordan's. The Nordtvedt parameter, accounting for a difference between the gravitational and inertial mass of a body caused by its self field, is a measure of the extent of the violation of the principle of equivalence ("Nordtvedt effect" [11]).

Considering together (3.11) and (3.23), it is then proved that under the assumptions a), b), and c) stated at the beginning of this section, the equation of motion (2.16) can be derived from the Lagrangian

$$\begin{aligned}
 L(a, \dot{a}, b, \dot{b}) &= m_A + m_B + \\
 &+ \frac{1}{2} m_A \left(1 + \frac{1}{4} \dot{a}^2 \right) \dot{a}^2 + \frac{1}{2} m_B \left(1 + \frac{1}{4} \dot{b}^2 \right) \dot{b}^2 + \\
 &+ \frac{m_A m_B}{8\pi R} \left[\frac{2\gamma+1}{2} (\dot{a}^2 + \dot{b}^2) - \frac{4\gamma+3}{\sqrt{2}} \underline{\dot{a}} \cdot \underline{\dot{b}} - \frac{(\underline{\dot{a}} \cdot \underline{R})(\underline{\dot{b}} \cdot \underline{R})}{2 R^2} \right] +
 \end{aligned}$$

$$\begin{aligned}
& + \frac{1-2\beta}{2(8\pi)^2} \frac{m_A m_B (m_A + m_B)}{R^2} + \frac{m_A^{(a)} m_B^{(a)}}{8\pi R} - \\
& - \left\{ \frac{1+2\gamma}{2} \frac{m_B}{8\pi} \frac{\underline{R} \times \dot{\underline{a}}}{R^3} - (1+\gamma) \frac{m_B}{8\pi} \frac{\underline{R} \times \dot{\underline{b}}}{R^3} \right\} \cdot \underline{S}_A + \\
& + \left\{ \frac{1+2\gamma}{2} \frac{m_A}{8\pi} \frac{\underline{R} \times \dot{\underline{b}}}{R^3} - (1+\gamma) \frac{m_A}{8\pi} \frac{\underline{R} \times \dot{\underline{a}}}{R^3} \right\} \cdot \underline{S}_B + \\
& + O(R^{-3}).
\end{aligned} \tag{3.25}$$

The first two terms are constants with respect to the dynamical variables $a, \dot{a}, b, \dot{b}$, and they play the role of integration constants. Their choice is made by requiring the correct newtonian limit.

The Lagrangian (3.25) generalizes the one obtained in the previous chapter, since it includes the PPN parameters β and γ , and allows the bodies to be spheroids in rigid rotation.

(4): The nature of the motion

In the previous section it was shown how the equation of motion for an extended body (2.16), under certain assumptions on the system (assumptions a), b), and c)), can be derived from the Lagrangian (3.25). In the present section this Lagrangian will be used in order to study the properties of the orbital motion of each body, and furthermore it will be shown, under an additional assumption, to provide the correct description even for the motion of the spin.

We begin with a remark on the definition of the center of mass of the system in the post-newtonian approximation; then the orbital motion is considered, using the equations for the perturbation of the orbital elements.

Finally, the rotational motion for each body is analysed, using the equation expressing the conservation of the total angular momentum of the system, and the assumption that the equation of motion for the spin is hamiltonian.

(4a): Remark on the center of mass condition for the system

In the post-newtonian approximation to fully conservative theories, the point defined by the vector

$$X_{\alpha} := \frac{\int \varrho^* x_{\alpha} \left(1 + \frac{v^2}{2} - \frac{U}{2} + \pi \right) d^3x}{\int \varrho^* \left(1 + \frac{v^2}{2} - \frac{U}{2} + \pi \right) d^3x} \quad (4.1)$$

moves with uniform velocity with respect to the global, harmonic coordinate system, as a consequence of the equations of motion for the whole system [15]. It can therefore be convenient to choose the point (4.1) as the origin of the global, PPN coordinate system, and to call it "center of inertial mass" of the system. It differs from the point defined by

$$\int \rho x_\alpha d^3x \quad (4.2)$$

To determine the deviation of (4.1) from (4.2), consider in (4.1) the usual decomposition of the velocities, of the potentials and of the position

$$\underline{V} = \underline{\dot{a}} + \underline{\dot{\xi}}; \quad U = u_a + U^{(a)}$$

(this holds for A; likewise for B), and the multipole expansion (3.5) of the external potential. Then, for $\underline{x} = 0$, we have

$$\begin{aligned} & m_A a_\alpha \left(1 + \frac{\dot{a}^2}{2} - \frac{1}{2} \frac{m_B}{8\pi R} \right) + m_B b_\alpha \left(1 + \dot{b}^2 - \right. \\ & \left. - \frac{1}{2} \frac{m_A}{8\pi R} \right) + \frac{1}{2} (\underline{\dot{a}} \times \underline{S}_A)_\alpha + \frac{1}{2} (\underline{\dot{b}} \times \underline{S}_B)_\alpha + \\ & + \frac{1}{2} \frac{m_B}{8\pi R^2} R_\beta \frac{I}{A} \alpha\beta - \frac{1}{2} \frac{m_A}{8\pi R^3} R_\beta \frac{I}{B} \alpha\beta = \quad (4.3) \\ & = O(R^{-3}) \end{aligned}$$

From (4.3) it follows

$$m_A a_\alpha + m_B b_\alpha = [\varepsilon^4, \varepsilon^2 \delta^2] \quad (4.4)$$

Then, using (4.4) in the post-newtonian terms of (4.3), and defining the total and reduced mass respectively as

$$m_0 := m_A + m_B \quad ; \quad \mu := \frac{m_A m_B}{m_0}$$

and the relative velocity

$$\underline{v} := \underline{\dot{R}} = \underline{\dot{a}} - \underline{\dot{b}}$$

we finally obtain

$$\begin{aligned} m_A a_\alpha + m_B b_\alpha = & \frac{1}{2} \frac{m_A - m_B}{m_0} R_\alpha \left(\mu v^2 - \frac{m_A m_B}{8\pi R} \right) - \\ & - \frac{1}{2} \left[\underline{v} \times \left(\frac{m_B}{m_0} \underline{S}_A - \frac{m_A}{m_0} \underline{S}_B \right) \right]_\alpha - \\ & - \frac{1}{2} \left(m_B \frac{I_A}{A} \alpha_\beta - m_A \frac{I_B}{B} \alpha_\beta \right) \frac{R_\beta}{8\pi R^3} + \\ & + O(R^{-3}) \end{aligned} \quad (4.5)$$

where I_A and I_B are (see 2.19) the tensors of inertia of A

and B, respectively.

From (4.5) it is clear that the point defined by the vector

$$m_A a_\alpha + m_B b_\alpha \quad (4.6)$$

and representing the newtonian center of mass, does not coincide with the point defined by (4.1) (the PPN center of inertial mass), unless the two bodies have identical masses, spins and tensors of inertia.

Furthermore the newtonian center of mass is accelerated with respect to the PPN one: the acceleration is however only periodic and non secular, that is the newtonian center of mass oscillates about a mean position.

In the following, the origin of the coordinates will be defined by the condition

$$m_A a_\alpha + m_B b_\alpha = 0 \quad (4.7)$$

As shown above, this point does not move uniformly, but this does not cause trouble, over time scales of the order of the orbital period.

Using (4.7), the Lagrangian (3.25) can then be rewritten as follows

$$\begin{aligned}
\mathcal{L} := \frac{L}{\mu} = & \frac{m_0}{\mu} + \frac{1}{2} v^2 + \frac{1}{8} \left(1 - \frac{3\mu}{m_0} \right) v^4 + \\
& + \frac{v^2}{2 \cdot 8\pi R} \left[(2\gamma+1) m_0 + \mu \right] + \frac{\mu}{2} \frac{(R \cdot v)^2}{8\pi R^3} + \\
& + \frac{1-2\beta}{2(8\pi)^2} \frac{m_0^2}{R^2} + \mathcal{N}^2 \frac{m_0}{8\pi R} - \\
& - \frac{1}{8\pi} \frac{R \times v}{R^3} \cdot \left[\left(\frac{1+2\gamma}{2} \frac{m_0}{m_A} + 1+\gamma \right) \underline{S}_A + \right. \\
& \left. + \left(\frac{1+2\gamma}{2} \frac{m_A}{m_B} + 1+\gamma \right) \underline{S}_B \right] + \\
& + O(R^{-3})
\end{aligned} \tag{4.8}$$

where the Nordtvedt ratio

$$\mathcal{N}^2 := \frac{m_A^{(a)} m_B^{(a)}}{m_A m_B} \tag{4.9}$$

has been introduced.

(4b): The orbital motion

Purpose of this subsection is to analyse the relativistic effects on the orbit, implied by (4.8).

Defining

$$\underline{A} := \left(\frac{1+2\gamma}{2} \frac{m_B}{m_A} + 1 + \gamma \right) \underline{S}_A \quad (4.10)$$

$$\underline{B} := \left(\frac{1+2\gamma}{2} \frac{m_A}{m_B} + 1 + \gamma \right) \underline{S}_B$$

and the spin-interaction vector potential

$$\underline{Z} := \frac{1}{8\pi R^3} \underline{R} \times (\underline{A} + \underline{B}) \quad , \quad (4.11)$$

we consider the following part of (4.8)

$$\mathcal{L}_1 = \frac{\underline{v}^2}{2} + \mathcal{N}^2 \frac{m_0}{8\pi R} + \underline{v} \cdot \underline{Z} \quad (4.12)$$

The corresponding equations of relative orbital motion are

$$\frac{d\underline{v}_\alpha}{dt} = - \frac{\mathcal{N}^2 m_0}{8\pi R^3} \underline{R}_\alpha + \underline{v} \times \nabla \times \underline{Z} \quad . \quad (4.13)$$

An equation similar to (4.13) was studied by Lense and Thirring [9] in the case of a test particle in geodesic motion around a spinning, massive body. In this case it was

$$\underline{S}_A = 0 \quad ; \quad m_A \ll m_B \quad ; \quad \gamma = 1 \quad .$$

Equation (4.13), derived independently of the assumption of geodesic motion, generalises the equation studied by Lense

and Thirring, allowing for rotation of both bodies, and non requiring m_A small compared to m_B .

In the rest of this subsection it will be established in which way the relative orbit is influenced by the rotation of the bodies.

Define

$$\mathcal{R} := \underline{v} \cdot \underline{z} \equiv - \frac{\underline{R} \times \underline{v}}{8\pi R^3} \cdot (\underline{A} + \underline{B}) \quad (4.14)$$

as the relativistic disturbing function in the Lagrangian (4.12), and introduce the six orbital elements:

- a major semi-axis of the keplerian ellipse;
- e eccentricity " " " " ;
- i inclination of the orbit with respect to the direction $\underline{A+B}$;
- Ω longitude of the ascending node with respect to the plane orthogonal to $\underline{A+B}$ (xy plane);
- $\tilde{\omega}$ longitude of periastron;
- ε mean anomaly of the epoch $t=0$ from the x-axis.

The origin of the coordinates is already defined by (4.7). The z-axis (see figure) is oriented as the vector $\underline{A+B}$ which is constant, to sufficient accuracy, while the x-axis is in the direction of an unspecified "equinox".

The equations for the perturbation of the orbital elements are, in the Gauss form ([4], Ch. 11),

$$\begin{aligned}
\frac{da}{dt} &= \frac{2}{n(1-e^2)^{1/2}} \left(P e \sin f + Q \frac{p}{R} \right) \\
\frac{de}{dt} &= \frac{(1-e^2)^{1/2}}{na} \left[P \sin f + Q (\cos u + \cos f) \right] \\
\frac{di}{dt} &= \frac{1}{na(1-e^2)^{1/2}} \frac{WR}{a} \cos u \\
\sin i \frac{d\Omega}{dt} &= \frac{WR \sin u}{na^2(1-e^2)^{1/2}} \\
\frac{d\tilde{\omega}}{dt} &= \frac{(1-e^2)^{1/2}}{nae} \left[-P \cos f + Q \left(\frac{R}{p} + 1 \right) \sin f \right] + \\
&\quad + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt} \\
\frac{dE}{dt} &= -2 \frac{R}{na^2} + \frac{e^2}{1+(1-e^2)^{1/2}} \frac{d\tilde{\omega}}{dt} + \\
&\quad + 2(1-e^2)^{1/2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt}
\end{aligned} \tag{4.15}$$

where the following auxiliary quantities have been used

$n := (m_{\odot}/8\pi a^3)^{1/2}$ is the mean motion;
 $p := a(1-e^2)$ is the semilatus rectum;
 f is the true anomaly from periastron;
 $u = f + \tilde{\omega} - \Omega$ is the true anomaly from the node;
 P, Q, W are the radial, tangential and
 respectively normal components of the
 perturbing acceleration associated to
 (4.14).

The components P , Q , and W will now be evaluated in order to obtain, via (4.15), the time variation of the orbital elements.

The relation between polar and cartesian coordinates is

$$x = R (\cos u \cos \vartheta - \sin u \sin \vartheta \cos i)$$

$$y = R (\cos u \sin \vartheta + \sin u \cos \vartheta \cos i)$$

$$z = R \sin u \sin i$$

from which one has

$$V_x = \frac{\dot{R}x}{R} - R\dot{u} (\sin u \cos \vartheta + \cos u \sin \vartheta \cos i)$$

$$V_y = \frac{\dot{R}y}{R} + R\dot{u} (-\sin u \sin \vartheta + \cos u \cos \vartheta \cos i)^{(4.16)}$$

$$V_z = \frac{\dot{R}z}{R} + R\dot{u} \cos u \sin i$$

From (4.13) it follows that in the frame (x, y, z) the components of the perturbing specific force are

$$\begin{aligned} F_\mu &:= (\underline{V} \times \nabla \times \underline{Z})_\mu = \varepsilon_{\mu\beta\alpha} \frac{V_\beta}{8\pi R^3} \left(\delta_{\alpha z} - 3 \frac{x_\alpha x_z}{R^2} \right) (\underline{A} + \underline{B})_z \\ &= \frac{|\underline{A} + \underline{B}|}{8\pi R^3} \varepsilon_{\mu\alpha\beta} V_\beta \left(3 \sin u \sin i \frac{x_\alpha}{R} - \delta_{\alpha z} \right) \end{aligned}$$

Calling X, Y, Z the components of $\underline{F}$ in the orthogonal frame, and making use of (4.16), we have

$$\begin{aligned} X &= \frac{|\underline{A} + \underline{B}|}{8\pi R^3} \left\{ R\dot{u} \left(3 \sin u \sin \vartheta \sin^2 i - \sin u \sin \vartheta + \right. \right. \\ &\quad \left. \left. + \cos u \cos \vartheta \cos i \right) + \dot{R} \left(\cos u \sin \vartheta + \sin u \cos \vartheta \cos i \right) \right\} \end{aligned}$$

$$Y = \frac{|A+B|}{8\pi R^3} \left\{ R \dot{u} \left(-3 \sin u \cos \vartheta \sin^2 i + \sin u \cos \vartheta + \right. \right. \\ \left. \left. + \cos u \sin \vartheta \cos i \right) - \dot{R} \left(\cos u \cos \vartheta - \sin u \sin \vartheta \cos i \right) \right\}$$

$$Z = \frac{|A+B|}{8\pi R^3} 3 R \dot{u} \sin u \sin i \cos i \quad .$$

Using the fact that

$$P = X \left(\cos u \cos \vartheta - \sin u \sin \vartheta \cos i \right) + Y \left(\cos u \sin \vartheta + \right. \\ \left. + \sin u \cos \vartheta \cos i \right) + Z \sin u \sin i$$

$$Q = -X \left(\sin u \cos \vartheta + \cos u \sin \vartheta \cos i \right) - Y \left(\sin u \sin \vartheta - \right. \\ \left. - \cos u \cos \vartheta \cos i \right) + Z \cos u \sin i$$

$$W = X \sin \vartheta \sin i - Y \cos \vartheta \sin i + Z \cos i \quad ,$$

we obtain

$$P = \frac{|A+B|}{8\pi R^4} \left(R^2 \dot{u} \cos i \right)$$

$$Q = -\frac{|A+B|}{8\pi R^3} \dot{R} \cos i$$

$$W = \frac{|A+B|}{8\pi R^4} \sin i \left(2 R^2 \dot{u} \sin u + R \dot{R} \cos u \right) \quad .$$

Further, considering the perturbations of the first order and taking advantage of the conservation of the angular

momentum, and of the formulae for the elliptic motion

$$l = R^2 \dot{u} = \text{const} \quad ; \quad \frac{1}{R} = \frac{1+e \cos f}{p} \quad ; \quad \dot{R} = \frac{R^2 \dot{f}}{p} e \sin f \quad ,$$

we finally write

$$P = \frac{|A+B| l \cos i}{8\pi R^4}$$

$$Q = - \frac{|A+B| l}{8\pi R^3 p} e \sin f \cos i$$

$$W = \frac{|A+B| l}{8\pi R^4} \left(2 \sin u + \frac{eR}{p} \sin f \cos u \right) \sin i \quad .$$

From (4.15) it follows

$$\frac{da}{dt} = 0$$

$$\frac{de}{dt} = \frac{(1-e^2)^{1/2}}{na} \cdot \frac{|A+B| l}{8\pi} \left[\frac{\sin f \cos i}{R^4} - \frac{e \sin f \cos i (\cos u + \cos f)}{R^3 p} \right]$$

$$\frac{di}{dt} = \frac{|A+B| l}{na^2 (1-e^2)^{1/2}} \frac{l \sin i}{8\pi R^3} \left(2 \sin u \cos u + \frac{eR}{p} \sin f \cos^2 u \right)$$

$$\sin i \frac{d\Omega}{dt} = \frac{|A+B| l}{na^2 (1-e^2)^{1/2}} \frac{l \sin i}{8\pi R^3} \left(2 \sin^2 u + \frac{eR}{p} \sin f \sin u \cos u \right)$$

$$\begin{aligned} \frac{d\omega}{dt} = & - \frac{\sqrt{1-e^2}}{nae} \frac{|A+B| l}{8\pi} \left[\frac{\cos i \cos f}{R^4} + \frac{e}{R^3 p} \left(\frac{R}{p} + 1 \right) \sin^2 f \cos i \right] \\ & + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt} \end{aligned}$$

$$\frac{d\varepsilon}{dt} = -\frac{2}{na^2} \frac{|A+B| \ell \cos i}{8\pi R^3} + \frac{e^2}{1+\sqrt{1-e^2}} \frac{d\tilde{\omega}}{dt} + 2\sqrt{1-e^2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt}.$$

For a non zero inclination angle "i", these expressions can be simplified as follows

$$\frac{de}{dt} = \frac{|A+B|}{8\pi \ell a} \cos i \sin f \dot{f}$$

$$\frac{di}{dt} = \frac{|A+B|}{8\pi \ell p} \dot{f} \cos u \left[2(1+e \cos f) \sin u + e \sin f \cos u \right] \sin i$$

$$\frac{d\Omega}{dt} = \frac{|A+B|}{8\pi \ell p} \dot{f} \sin u \left[2(1+e \cos f) \sin u + e \sin f \cos u \right]$$

$$\frac{d\tilde{\omega}}{dt} = -\frac{|A+B| \cos i}{8\pi e p \ell} \dot{f} \left[2e + (1+e^2) \cos f \right] + 2 \sin^2 \frac{i}{2} \frac{d\Omega}{dt}$$

$$\frac{d\varepsilon}{dt} = -\frac{2|A+B|}{8\pi na^2 p} (1+e \cos f) \cos i \dot{f} + \frac{e^2}{1+\sqrt{1-e^2}} \frac{d\tilde{\omega}}{dt} + 2\sqrt{1-e^2} \sin^2 \frac{i}{2} \frac{d\Omega}{dt},$$

where a small error in $\tilde{\omega}$ appearing in [9] has been corrected. Integrating in time, and keeping constant the orbital elements in the right hand side, we have

$$\Delta a = 0$$

$$\Delta e = -\frac{|A+B|}{8\pi \ell a} \cos f$$

$$\Delta i = - \frac{|A+B|}{8\pi \ell p} \sin i \left[\cos 2u + e \cos f \cos^2 u \right]$$

$$\Delta \Omega = \frac{|A+B|}{8\pi \ell p} \left[u - \frac{1}{2} \sin 2u + e \left(\sin f - \frac{1}{2} \sin 2u \cos f \right) \right]$$

$$\Delta \tilde{\omega} = - \frac{|A+B|}{8\pi e p \ell} \cos i \left[2e f + (1+e^2) \sin f \right] + 2 \sin^2 \frac{i}{2} \Delta \Omega$$

$$\Delta \varepsilon = - \frac{2|A+B|}{8\pi m a^2 p} \cos i \left(f + e \sin f \right) + \frac{e^2}{1+\sqrt{1-e^2}} \Delta \tilde{\omega} + 2 \sqrt{1-e^2} \sin^2 \frac{i}{2} \Delta \Omega$$

where a small error in Δi appearing in [9] has been corrected.

Recalling that

$$u = nt + \text{period} ,$$

and that

$$1 = na^2(1-e^2)^{1/2} ,$$

we thus obtain as secular, first order perturbations

$$\frac{d\Omega}{dt} = \frac{1}{8\pi} \frac{|A+B|}{a^3(1-e^2)^{3/2}}$$

$$\frac{d\varepsilon}{dt} = \frac{d\tilde{\omega}}{dt} = \frac{1}{8\pi} \frac{|A+B|}{a^3(1-e^2)^{3/2}} (1-3\cos i) \quad (4.17)$$

$$\frac{de}{dt} = \frac{di}{dt} = \frac{da}{dt} = 0 .$$

These equations provide a generalization of the Lense-

Thirring effects. Now infact the spins and the masses of both the bodies are responsible for the variation of the orbital elements, and no assumption about the motion (i.e. geodesic world line) is made.

It might be noted that in the case $\underline{A} + \underline{B} = 0$, the Lense-Thirring effects are zero, although both bodies may be rotating. This clearly happens for equal and opposite spins, and for equal masses.

Finally, to conclude the analysis of the orbital motion, let us recall that the spin-independent part of the Lagrangian (4.8) gives rise to the well-known Einstein's precession of periastron, which in the PPN formalism [15], for an orbital period P_0 , is

$$8 \pi \dot{\tilde{\omega}}_E = (2 - \beta + 2\gamma) \frac{2 \pi m_0}{P_0 a (1 - e^2)}$$

Whence, using c.g.s. unities, the total relativistic periastron precession from a fixed x-axis is

$$\begin{aligned} \dot{\tilde{\omega}} = \frac{G}{c^2} \left[(2 - \beta + 2\gamma) \frac{2 \pi m_0}{P_0 a (1 - e^2)} + \right. \\ \left. + \frac{|\underline{A} + \underline{B}|}{a^3 (1 - e^2)^{3/2}} (1 - 3 \cos i) \right] \end{aligned} \quad (4.18)$$

We conclude here the analysis of the possible relativistic effects on the orbit, and turn to the relativistic effect in the intrinsic motion of each body.

(4c): The rotational motion (§)

In this subsection the newtonian expressions for the orbital angular momentum of a two body system, and for the spin of each body will be generalized by introducing the corresponding definitions in the post-newtonian framework.

It will further be shown that the equation of motion for the spin defined in this way represents a precession of the spin about the direction of the orbital angular momentum.

The definitions of the spin and of the orbital angular momentum are obtained by decomposition of the total angular momentum

$$L_\alpha := \varepsilon_{\alpha\beta\gamma} \int_{A+B} x^\beta \pi^\gamma d^3x \quad (4.19)$$

which is conserved because of the local equations of motion (2.2). It can in fact be proved that

$$\int_{A+B} \sigma_{[\alpha\beta]} d^3x = \int_{A+B} x_{[\alpha} f_{\beta]} d^3x .$$

Considering, for the external part of the potentials occurring in π_α , a multipole expansion truncated at the monopole term, the total angular momentum (4.19) can be decomposed into the following four parts:

(§) The results of this subsection have been obtained in collaboration with Dr. N. Spyrou.

$$\underline{L}_\alpha = \underline{J}_\alpha + \mathcal{S}_A^\alpha + \mathcal{S}_B^\alpha + K_\alpha + O(R^{-2}) \quad (4.20)$$

where

$$\underline{J} := \mu \underline{R} \times \underline{V} \left\{ 1 + \frac{V^2}{2} \left(1 - \frac{3\mu}{m_0} \right) + \frac{1}{8\pi R} [(1+2\gamma)m_0 + \mu] \right\} \quad (4.21)$$

$$\mathcal{S}_A^\alpha := \left(1 - \gamma \frac{m_B}{8\pi R} \right) \dot{S}_A^\alpha - \frac{1}{2} \dot{a}_\alpha (\underline{\dot{a}} \cdot \underline{S}_A) , \quad (4.22)$$

and

$$\begin{aligned} \dot{S}_A^\alpha := & \left(1 + \dot{a}^2 + \frac{3\gamma+1}{8\pi R} m_B \right) \epsilon_{\alpha\beta\gamma} \int_A \mathcal{S}^* \dot{S}_\beta \left[1 + \right. \\ & + \frac{\dot{S}^2}{2} - \mu_a + \pi + p/\mathcal{S}^* \left. \right] \dot{S}_\gamma + \frac{1}{2} \left(\bar{w}_{(a)\gamma} - \bar{u}_{(a)\mu} \gamma^\mu \right) + \\ & + (2\gamma+2) \left(\dot{S}_\gamma \mu_a - \bar{w}_{(a)\gamma} \right) \Big] d^3x . \end{aligned} \quad (4.23)$$

Analogous definitions for $\underline{S}_B$ and $\dot{S}_B$ are obtained from (4.22), (4.23) simply interchanging the labels of A and B.

Finally

$$\begin{aligned} \underline{K} := & \frac{\mu}{8\pi R^3} \left[\left(\frac{2\gamma+1}{2} \frac{m_B}{m_A} + 1 + \gamma \right) \underline{R} \times (\underline{R} \times \underline{S}_A) + \right. \\ & \left. + \left(\frac{2\gamma+1}{2} \frac{m_A}{m_B} + 1 + \gamma \right) \underline{R} \times (\underline{R} \times \underline{S}_B) \right] . \end{aligned} \quad (4.24)$$

Following Spyrou 13, the vector $\underline{J}$ in (4.21) is defined to be the post-newtonian orbital angular momentum: it is

parallel to the newtonian one, but its magnitude is different.

The vector $\underline{S}_A$ is defined in terms of $\underline{S}_A$ by (4.22). We interpret the S_A^α as the components of the spin of the body A in the global, PPN coordinate frame, and S_A^α as the components of the spin of A in the comoving tetrad, i.e. the one attached to the center of mass of A and having the axis parallel to those of the inertial, global PPN system of reference.

The interpretation of S_A as spin of A is in accordance with the proposal by Spyrou [13] and comes from the following argument: as in newtonian theory the spin of a body is constructed by integration of the external product of $\underline{\xi}$ with the newtonian "self-momentum density" $\varrho \underline{\dot{\xi}}$, so in the post-newtonian framework the spin of a body is essentially the integral of the external product of $\underline{\xi}$ with the post-newtonian self-momentum density

$$\varrho^* \left[\left(1 + \frac{\dot{\xi}^2}{2} - \mu_a + \pi + p/\varrho^* \right) \dot{\xi}_\alpha + (2\gamma+2)(\dot{\xi}_\alpha \mu_a - \bar{w}_{(a)\alpha}) + \frac{1}{2} \left(\bar{w}_{(a)\alpha} - \bar{\mu}_{(a)} r_\alpha \cdot r \right) \right] \quad (4.25)$$

(see also eq. 2.18).

Expression (4.23) differs from the definition already given in [13] by the coefficient

$$1 + \dot{a}^2 + (3\gamma+1) \frac{m_B}{8\pi R}$$

which depends on the orbital velocity of A and the gravitational

potential of B. In [13] this coefficient was considered only in order of magnitude, while the present analysis goes one step further and takes it into account in the equations. The relation (4.22) between the components S_A^α of the spin in the comoving tetrad and the components S_A^α of the spin in the inertial frame, is already well-known from the theory of the relativistic spin precession of an orbiting gyroscope (see [8] ch. 40, or [14] ,ch.9). In this framework the derivation of (4.22) crucially depends on the covariant center of mass condition, that is, if S and $\dot{a}$ are the four-dimensional spin and, respectively, velocity of a body, then

$$S^m \dot{a}_m = 0$$

covariantly defines the center of mass.

Finally we note that during an orbital period the vector $\underline{K}$ in (4.24) oscillates about a mean position, hence experiencing no secular change in magnitude and/or direction, at least in this approximation. On the time scale of an orbital period the vector $\underline{K}$ can then be considered constant.

Since the total angular momentum (4.19) is conserved, differentiating (4.20) with respect to the time and treating $\underline{K}$ as a constant, we obtain

$$-\dot{J}_\alpha = \frac{d}{dt} \left(S_A^\alpha + S_B^\alpha \right) \quad (4.26)$$

(hereafter the term in (4.20) expressing the fall-off with the distance will be, for simplicity, omitted).

The equation of motion for the orbital angular momentum is already known from the orbital analysis: the absolute value of $\underline{J}$ is constant, while its orientation in space changes, because of the generalized Lense-Thirring effect (4.17) in the longitude of the ascending node. The evolution of $\underline{J}$ is given by the following equation

$$\dot{\underline{J}} = \underline{\Omega} \times \underline{J} \quad (4.27)$$

where

$$\underline{\Omega} := \frac{1}{8\pi R^3} \left[\left(\frac{2\gamma+1}{2} \frac{m_B}{m_A} + 1 + \gamma \right) \underline{S}_A + \left(\frac{2\gamma+1}{2} \frac{m_A}{m_B} + 1 + \gamma \right) \underline{S}_B \right] \quad (4.28)$$

(Note that at this order of accuracy there is no distinction in (4.28) between $\underline{S}$ and $\underline{J}$).

From (4.26) and (4.27) it follows

$$\frac{d}{dt}(\underline{S}_A + \underline{S}_B) = \underline{\Omega}_A \times \underline{S}_A + \underline{\Omega}_B \times \underline{S}_B \quad (4.29)$$

with

$$\begin{aligned} \underline{\Omega}_A &:= \mu \frac{\underline{R} \times \underline{V}}{8\pi R^3} \left[\frac{2\gamma+1}{2} \frac{m_B}{m_A} + 1 + \gamma \right] \\ \underline{\Omega}_B &:= \mu \frac{\underline{R} \times \underline{V}}{8\pi R^3} \left[\frac{2\gamma+1}{2} \frac{m_A}{m_B} + 1 + \gamma \right] \end{aligned} \quad (4.30)$$

In order to derive the equation of motion for each spin separately, we assume that the equation of motion for each body can be derived from a Hamiltonian H . In the notation of the Poisson brackets we then have

$$\frac{d}{dt} \underline{S}_i = [\underline{S}_i, H] \quad (i = A, B) \quad (4.31)$$

where for each body the dynamical variables are the Euler angles, and the components S of the spin in the comoving tetrad are the conjugate momenta. We further assume that H contains no term which simultaneously depends on the canonical set of variables of A and B , but it may depend parametrically on the orbit variables.

Writing (4.29) in the form

$$\begin{aligned} \underline{\Omega}_A \times \underline{S}_A + \underline{\Omega}_B \times \underline{S}_B &= \\ &= [\underline{S}_A, H] + [\underline{S}_B, H] \end{aligned} \quad (4.32)$$

we obtain the equation of motion for $\underline{S}_A$ or $\underline{S}_B$, by forming the Poisson bracket of (4.32) with $\underline{S}_A$ or, respectively, $\underline{S}_B$, and using the following identities:

$$[\underline{S}_A, \underline{S}_B] = 0$$

$$[S_A^\alpha, S_A^\beta] = \epsilon_{\alpha\beta\gamma} S_A^\gamma \quad (\text{likew. for } \underline{S}_B)$$

$$[\underline{S}_A, [\underline{S}_B, H]] = [\underline{S}_B, [\underline{S}_A, H]] = 0$$

$$[\underline{S}_A, \underline{S}] = [\underline{S}_B, \underline{S}] = 0$$

$$\epsilon_{\alpha\beta\mu} [(\underline{\Omega}_A \times \underline{S}_A)_\alpha, S_{A\beta}] = (\underline{\Omega}_A \times \underline{S}_A)_\mu$$

(likewise for B), and using the Poisson identity for any three quantities A, B, C:

$$[A, [B, C]] + [C, [A, B]] + [B, [C, A]] = 0$$

The result is

$$\frac{d}{dt} \underline{S}_A = \underline{\Omega}_A \times \underline{S}_A \quad (4.33)$$

$$\frac{d}{dt} \underline{S}_B = \underline{\Omega}_B \times \underline{S}_B \quad (4.34)$$

Equations (4.33) and (4.34) show that, as a consequence of the definitions (4.21), (4.22), (4.23) and of the equation (4.26) expressing the balance of the various angular momenta in the system, the spin of each body precesses about the direction of the orbital angular momentum, to which both $\underline{\Omega}_A$ and $\underline{\Omega}_B$ are parallel.

We notice that in the previous sections the spin never needed to be defined at the post-newtonian accuracy: it always

appeared in terms of post-newtonian order, so that the newtonian definition for the spin was sufficient.

The assumptions made on the existence of a Hamiltonian and on its structure are not too restrictive. A Hamiltonian satisfying the requirements of absence of a coupled term in the two sets of canonical variables is obtained by Legendre transformation from the Lagrangian (3.25), which in fact does give rise to the same equations (4.33), (4.34).

Therefore the Lagrangian (3.25), constructed upon the translational equations for two perfect fluid, extended bodies, describes correctly also the rotational motion of such bodies, if the assumption made above is correct.

Equations (4.33), (4.34) are identical to those obtained by Barker and O'Connell [1], Hari Dass and Cho [6] and by Börner, Ehlers and Rudolph [3]. Barker and O'Connell proved that in the framework of Gupta's quantum theory of gravitation, spin $1/2$ particles undergo the relativistic spin precession. Hari Dass and Cho, using semiclassical arguments, proved that the precession was indeed independent of the spin of the particles. Börner, Ehlers and Rudolph proved that the spin precesses as a consequence of the assumptions of Fermi-Walker transport of the spin along the world-line of the center of mass, and of geodesic motion of the center of mass.

In [3] it was moreover emphasized the need for a clear connection between the local equations of the post-newtonian,

perfect fluid hydrodynamics on the one hand, and the behavior of the spin of a perfect fluid, extended body on the other hand.

Trying to establish such connection was the motivation for this subsection^(§).

(§)

Mc Crea and O'Brien [10] gave a definition for the spin of a non perfect fluid, extended body, and proved that it undergoes a relativistic spin precession. They obtained a precessional frequency which differs from the one derived here and in [1] , [3] , [6] by an additional term. Its presence is due to the internal stresses associated to a non perfect fluid energy-momentum tensor, and it is non negligible in comparison with the other terms, in general.

(5): Discussion

In the framework of the PPN approximation to fully conservative theories of gravity, an approach has been presented aimed to provide a relativistic equation of motion for perfect fluid, extended bodies. The derivation of such equation is based on the reformulation (2.14) of the spatial part of the hydrodynamical equations (2.1). As in chapter 3, we stress that the two formulations are equivalent to each other, modulo terms of post-post-newtonian order. A post-newtonian equation of motion for an extended body, eq. (2.16), was then obtained. For every body of the system, the assumptions of symmetry under rotations about an axis through the body, and across a plane orthogonal to the axis, are essential in order to fix a reference point inside each body; its overall motion is then the motion of the center.

In Newtonian theory no internal symmetry is required in order to define the center of mass; an analogous, general condition for the definition of a center of mass in the post-newtonian approximation has not yet been rigorously formulated.

In the newtonian theory, it is precisely the center of mass condition which determines the vanishing of the part of the newtonian potential falling-off as R^{-2} : that is, decomposing the potential into its principal parts

$$U = U_0 + U_1 + U_2 + \dots + U_n + \dots$$

(with U_n falling-off as R^{-n-1}), $U_1 = 0$ if the center of mass condition holds. This also means that in the Lagrangian for the classical two-body problem, terms proportional to R^{-2} are absent. Looking at (3.23) it is seen that it is not so in the relativistic Lagrangian, where a variety of terms proportional to R^{-2} are present, in competition to the classical ones falling-off as R^{-3} and responsible of quadrupole interaction. For bodies with small quadrupole moment, as for instance neutron stars, there is in principle the possibility of resolving the relativistic effects in both the orbital and intrinsic motion, out of the newtonian ones.

An excellent example is given by the "binary pulsar" [7], a two body system composed by a neutron star (the pulsar itself) and by a compact companion (probably either a neutron star, or a white dwarf or a black hole).

A first post-newtonian effect is that the point defined by

$$m_A \underline{a} + m_B \underline{b} \quad (5.1)$$

does not move uniformly with respect to a distant, inertial observer. In subsection (4a) it was shown that, in the PPN framework, the uniformly moving point had coordinates differing from (5.1) by a quantity of post-newtonian order, whose dependence on the first few multipole moments of the bodies is given in (4.5). Defining the mass-ratio

$$x := \frac{m_B}{m_A}$$

we have for the leading term in (4.5) in c.g.s. unities

$$|\underline{a} + x \underline{b}| = \frac{R}{2} \left| \frac{v^2}{c^2} \frac{x(1-x)}{(1+x)^2} - \frac{G m_A}{c^2 R} \frac{x(1-x)}{1+x} \right|.$$

For the binary pulsar

$$(v/c)^2 \sim 10^{-6} \sim Gm/Rc^2$$

so that

$$|\underline{a} + x \underline{b}| \sim R \frac{x^2 |1-x|}{(1+x)^2} \quad (5.2)$$

with $R \sim 10^{11}$ cm.

From (5.2) it follows that the point defined in (5.1) is less than 1Km distant from the post-newtonian center of inertia(4.1).

The numerical values of the remaining terms in (4.5) drop rather quickly: the contribution to (5.2) from the spin-dependent part is maximal for

$$\underline{S}_A = - \underline{S}_B ; \underline{v} \perp \underline{S}$$

and it is roughly of order $vS/mc^2 \sim 1\text{cm}$ for a period of $\sim 10\text{msec}$.

The precession of periastron has been measured by Taylor and collaborators [12] with an accuracy of 0.2%

$$\dot{\omega} = 4.226 \pm .002 \text{ deg/yr.}$$

where the longitude of periastron has been measured from the ascending node rather than from a fixed "equinox":

$$\omega := \varpi - \Omega .$$

In fact the timing analysis of the pulses [2] , [5] used for computing the orbital elements is independent of Ω .

Apart from newtonian effects, the possibility of a secular change in Ω exists in connection with the generalized Lense-Thirring effects discussed in subsection (4b). It should be remarked that in this case the orbit was not oriented with respect to the line of sight, but with respect to the direction $\underline{A} + \underline{B}$ defined in (4.10). This is not a fixed direction in space because of the spin precession (subsect. 4c). Since however the latter is once again of relativistic nature, it must be taken into account only in the analysis of the second order relativistic perturbations in Ω .

Consider also $\underline{A} + \underline{B}$ as a fixed direction: the orbital plane rotates about it with an angular velocity given by the first of (4.17). For $\gamma = 1$

$$\dot{\Omega} = \frac{G}{c^2} \frac{|(2 + \frac{3}{2}\kappa)\underline{S}_A + (2 + \frac{3}{2}\kappa^{-1})\underline{S}_B|}{a^3(1-e^2)^{3/2}} \quad (5.3)$$

which is maximized for parallel spins. Assuming then $\underline{S}_A // \underline{S}_B$ and setting

$$|\underline{S}_B| = \alpha |\underline{S}_A|$$

for $x=1$ (5.3) takes the form

$$\dot{\Omega} = \left(\frac{2\pi}{c P_o} \right)^2 \cdot \frac{7}{2} (1 + \alpha) \frac{|\underline{S}|}{m_o (1 - e^2)^{3/2}}$$

where P_o is the orbital period and $m_o = m_A + m_B$.

For an application to the binary pulsar,

$$\begin{array}{ll} P_o = 2.79 \times 10^4 \text{ sec} & e = 0.62 \\ m_o = 2.85 m_\odot & P = 5.9 \times 10^{-2} \text{ sec} \\ \alpha = 1 \text{ (say)} & (P = \text{pulsar period}) \end{array}$$

Furthermore, calling I the moment of inertia of the pulsar,

$$|\underline{S}| = \frac{2\pi I}{P} \sim 10^{47} I_{45} \text{ g cm}^2 \text{ s}^{-1}$$

$$(I_{45} = I/10^{45}).$$

Then

$$\dot{\Omega} = 2.5 \times 10^{-5} I_{45} \text{ deg/yr.} \quad (5.4)$$

From (4.17) it follows that $\dot{\omega}$ is of the same order of magnitude as $\dot{\Omega}$, that is, for $I_{45} \sim 1$, at least two orders of magnitude below the present level of accuracy. The second term in the law for the relativistic precession (4.18) should then be excluded as reason for the 10% disagreement [12] in the value of the total mass, as

measured from the relativistic periastron precession on the one hand, and from the combined measurements of first and second order Doppler shift, and the change in orbital period- because of emission of gravitational radiation- on the other hand.

Relativistic spin precession (subsect. 4c) appears particularly interesting, especially for testing pulsar models.

From (4.30) it follows that the average precessional frequency is

$$\langle \Omega_A \rangle = \frac{G^{3/2}}{c^2} \mu \left(\frac{2\gamma+1}{2} \frac{m_B}{m_A} + 1 + \gamma \right) \frac{\sqrt{m_0}}{a^{5/2}(1-e^2)}$$

That is, for $\gamma=1$, $\Omega_A \approx 1$ degree per year. In the "hollow cone" model, this precession would produce an increasing or decreasing in the mutual distance of the peaks in the integrated pulse-profile, depending on whether the line of sight moves toward or, respectively, away from the magnetic axis.

Taylor's last observations [12] appear qualitatively in favour of the increase of the mutual distance between the peaks (broadening of the pulse profile).

More quantitative, even model-dependent statements should however be formulated, in view of future availability of more data, possibly including polarization.

Appendix

We prove that if assumptions a) and b) of sect. 3 hold, then the quantity $\underline{D}$ defined in (2.18) is of order

$$D_\alpha = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (\text{A.1})$$

Proof

It will be shown that $\underline{D}$ vanishes in the local, comoving coordinates $(\tilde{\xi}_\alpha, \tilde{t})$ by virtue of a), b), and that this property is preserved by a Lorentz transformation which relates the comoving coordinate system to the global system.

Consider the quantity

$$\int_A \varrho^* \xi_\alpha d^3x \quad (\text{A.2})$$

where

$$\varrho^* := \sqrt{-g} u^0 \varrho$$

is the first component of a vector density.

Define the invariant mass-element

$$dm := \varrho \sqrt{-g} d^3x$$

Assumption a) implies that in the comoving coordinates $(\tilde{\xi}_\alpha, \tilde{t})$

$$\int_A \tilde{\varrho}^* \tilde{\xi}_\alpha d^3 \tilde{\xi} = \int_A d\tilde{m} \tilde{\xi}_\alpha \tilde{u}^0 = 0 \quad (\text{A.3})$$

A Lorentz transformation relating the two local coordinate systems $(\tilde{\xi}_\alpha, \tilde{t})$ and $(\bar{x}_\alpha, \bar{t})$ brings a factor in front of the integral. Hence (A.3) holds in the inertial coordinates $(\bar{x}, \bar{t})$, and then also in the global coordinates $(\underline{x}, t)$, up to terms of second post-newtonian order

$$\int_A \varrho^* \xi_\alpha d^3 x = [\varepsilon^4 \delta, \varepsilon^2 \delta^3] \quad (\text{A.4})$$

Since ϱ^* satisfies the three dimensional law of conservation of the baryonic number, then

$$\frac{d}{dt} \int_A \varrho^* \xi_\alpha d^3 x = \int_A \varrho^* \dot{\xi}_\alpha d^3 x$$

(A.4) then implies

$$\int_A \varrho^* \dot{\xi}_\alpha d^3 x = [\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] \quad (\text{A.5})$$

It follows that (2.18) reduces to

$$\begin{aligned} D_\alpha = \int_A \varrho^* \left[\left(\frac{\dot{\xi}^2}{2} - \mu_\alpha + \pi + \frac{p}{\varrho^*} \right) \dot{\xi}_\alpha + (2\gamma+2) (\dot{\xi}_\alpha \mu_\alpha - \right. \\ \left. - \bar{w}_{(\alpha)}^\alpha) + \frac{1}{2} \left(\bar{w}_{(\alpha)}^\alpha - \bar{u}_{(\alpha)} \beta \cdot \alpha \beta \right) \right] d^3 x + (\text{A.6}) \\ \left[\varepsilon^6, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4 \right] \end{aligned}$$

Equation (A.6) is entirely of post-newtonian order, and then invariant under a first order Lorentz transformation (or Galilei transformation)

$$D_{\alpha} = \tilde{D}_{\alpha} + [\varepsilon^{\prime}, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] .$$

Furthermore (A.6) is "odd" in the internal velocity $\underline{\dot{\xi}}$, and therefore in $\underline{\xi}$, because of assumption b) (see eq. 3.1). It follows that $\underline{D}$ can be invariant under the symmetry group only if zero. Hence

$$D_{\alpha} = [\varepsilon^{\prime}, \varepsilon^4 \delta^2, \varepsilon^2 \delta^4] .$$

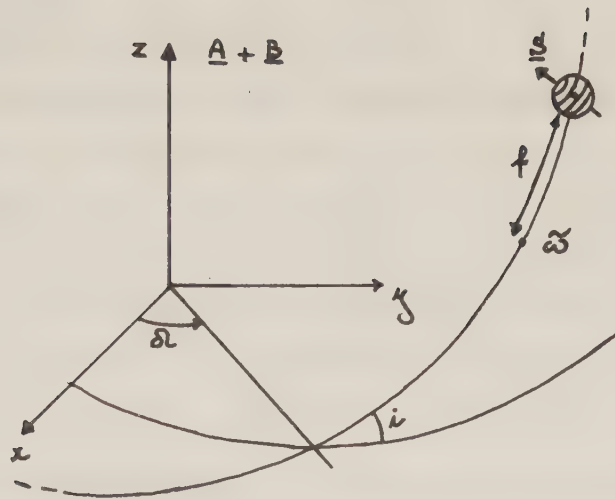


Figure: Orbit-orientation in space.

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Lebenslauf

Als Sohn des Gutsbesitzers Guido Caporali und seiner Ehefrau Giuliana, geb. Gagliardo, wurde ich am 16. Januar 1952 in Padua (Italien) geboren. Nachdem ich im Jahre 1970 den Besuch des humanistischen Gymnasiums in Padua mit der "Maturità" abgeschlossen hatte, begann ich an der dortigen Universität im Fachbereich Physik mein Studium. 1975 erhielt ich auf Grund einer Dissertation über die Struktur rotierender Flüssigkeiten in der Allgemeinen Relativitätstheorie den Grad eines "Dottore in Fisica". Im Jahre 1976 habe ich am Institut für Physik in Padua ohne Bezahlung meine Untersuchungen über rotierende Flüssigkeiten fortgesetzt. Seit dem Jahre 1977 bin ich als Doktorand von Prof. Jürgen Ehlers am Max Planck Institut für Physik und Astrophysik in München tätig.

